Efficient Algorithms on almost special Graph Classes or How to create new problems for algorithmic research?

Venkatesh Raman

(Retd) IMSc Chennai, currently honorary professor at IIT Palakkad

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Graph theorists study special classes of graphs like

- Trees, Forests
- Chordal Graphs
- Interval Graphs
- Circular Arc Graphs
- Perfect Graphs
- Bipartite Graphs
- Proper Interval Graphs
- Unit Interval Graphs
- Planar Graphs
- Cliques, Cluster Graphs (where each connected component is a clique)
- ....

and study (optimization) parameters like

- Maximum Clique
- Maximum Independent Set
- Minimum Dominating Set
- Chromatic Number
- Longest Path
- Longest Cycle
- Minimum Vertex Cover
- ....

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- Some of these optimization parameters/problems are solvable in polynomial time in these special classes of graphs (though *NP*-hard in general graphs).
  For example,
  - in the class of Chordal Graphs, we can determine a maximum clique, or independent set or chromatic number in polynomial time.
  - In Bipartite graphs, we can find a minimum vertex cover in polynomial time.
  - In Trees, we can find a minimum dominating set or a vertex cover in polynomial time.

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for appropriate definition of almost and appropriate definition of efficiency ....

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  Let us call such a graph almost special. Here *special* could be chordal or bipartite or forest ...

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- Recognition: Is the given graph almost special?
- Optimization: In the given almost special graph, how fast can we solve .... — optimization problem like VERTEX COVER or CLIQUE or CHROMATIC NUMBER?

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- $P : G \to N$ – a graph parameter

Some Algorithmic assumptions

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This talk: Recognition and Optimization in $\Pi + k$ graphs.

Recognizing $\Pi + k$ graphs.

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- So an algorithm with time Polynomial($|V(G)|, k$) is unlikely.

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  There is a notion of (W-)hardness in this paradigm as well.

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Some Recent Directions

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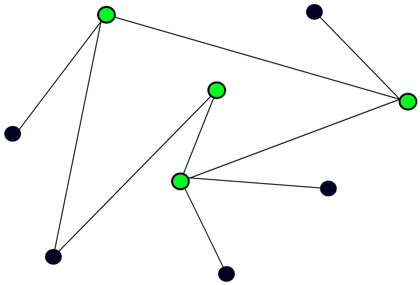
  Then recognizing G in $\Pi + k$ graphs is FPT. (Jacob et al. 2023, 2024)
  Corollary Are there k vertices whose removal makes every component a clique or a split graph? (FPT).

Optimization Problems in $\Pi + k$ graphs

Vertex Cover in a graph $G(V, E)$ – A subset S of vertices such that for every $(i, j) \in E$, i or j (or both) is in S.



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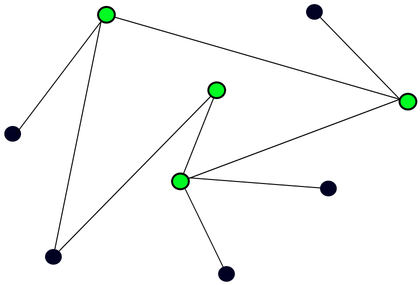

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- Degree 1 Preprocessing Rule For every pendant vertex x, include its unique neighbor in the solution and delete it. Delete all isolated vertices.
- Question What happens if we apply the rule on a tree (connected graph with no cycles)?

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- Replace forest by your favourite special graph class (Chordal, Bipartite, Planar, ...) and
  replace vertex cover by your favourite parameter (clique, independent set, dominating set, chromatic number) which is polynomial time solvable in that class
  you have several new problems ...

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- Guess the intersection Y of solution with S, delete Y. (Number of guesses is 2^k).
- Pick $N(S \setminus Y) \cap (V \setminus S)$ into the solution, delete them
- solve the (optimum) VC problem in the remaining forest.

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- **Theorem** If Π is a hereditary class of graphs where VERTEX COVER is solvable in polynomial time, then VERTEX COVER in $\Pi + k$ graphs has an FPT algorithm.

Dominating Set, Chromatic Number, Feedback Vertex Set in *Forest* $+ k$ graphs

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- Any graph in *Forest* $+ k$ class has treewidth at most $k + 1$, and so FVS, Dominating Set, Chromatic Number have *FPT* algorithm.
- **Open?** Are there algorithms without using treewidth machinery (like what we had for vertex cover)?

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- *Solving $P(G)$ without finding the modulator* – Determine if G is not Chordal$+k$, or solve your parameter in the graph in FPT time (Jacob et al. 2022).

- Algorithms in $\Pi + ke$ graphs (edge addition, deletion or contraction)
- What if S, the modulator is not given?
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- A lot more ...

More problems

- Take your favourite graph class that is recognizable in polynomial time.
- Pick a parameter that can be solved in that graph class in polynomial time.
- Now address the recognition and the parameter problem in graphs that are k away from the graph class in the paradigm of parameterized complexity.

Thank You