Barrier resilience problems and crossing numbers of graphs

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A bit about me

A bit about me

Outline

Two independent parts

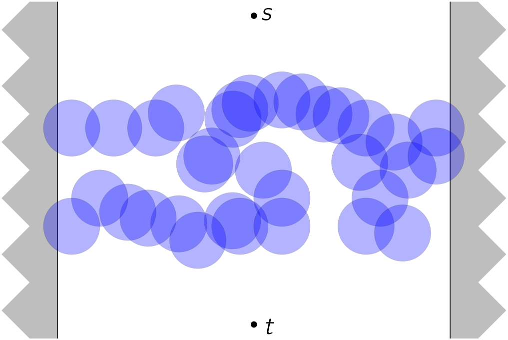
- Barrier resilience
- Crossing numbers of graphs

In common:

- Geometry
- Graph Theory
- Simple to state, interesting for general audience
- I have worked on them, I have something to explain

Original barrier resilience problem

[Kumar, Lai, Arora 2005], [Bereg, Kirkpatrick 2009]



rectangle

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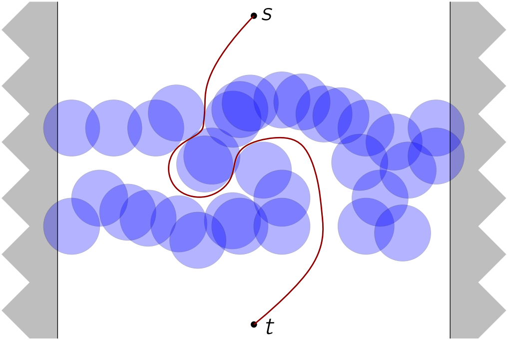

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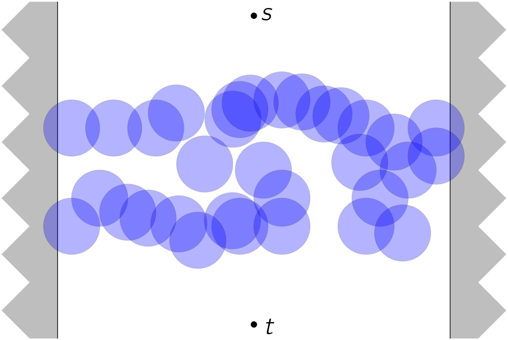

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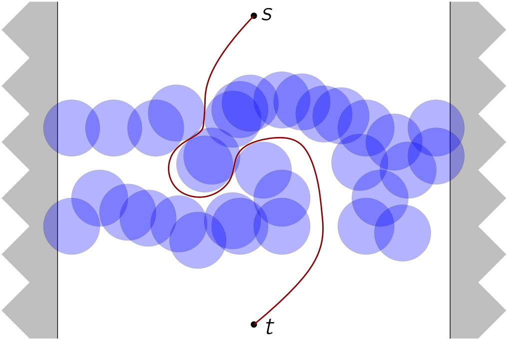

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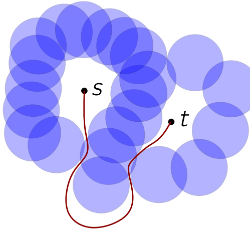

annulus

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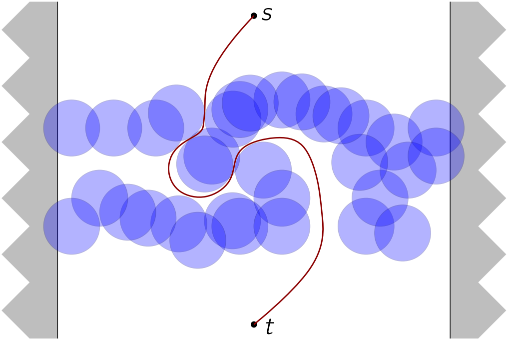

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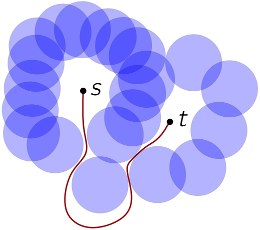

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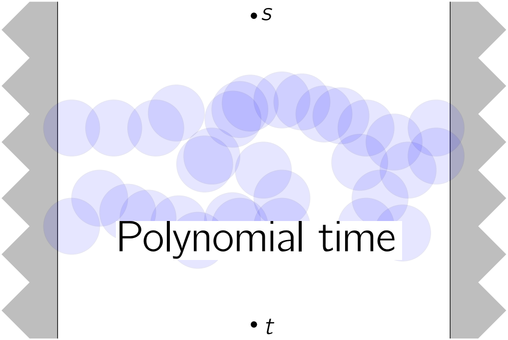

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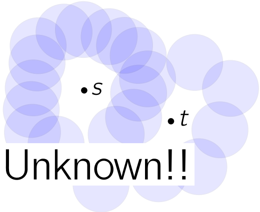
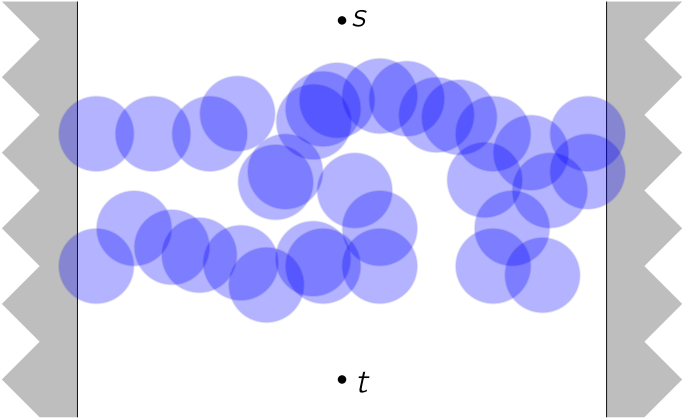


annulus

Barrier resilience - Rectangular domain in P

[Kumar, Lai, Arora 2005] via Menger's theorem



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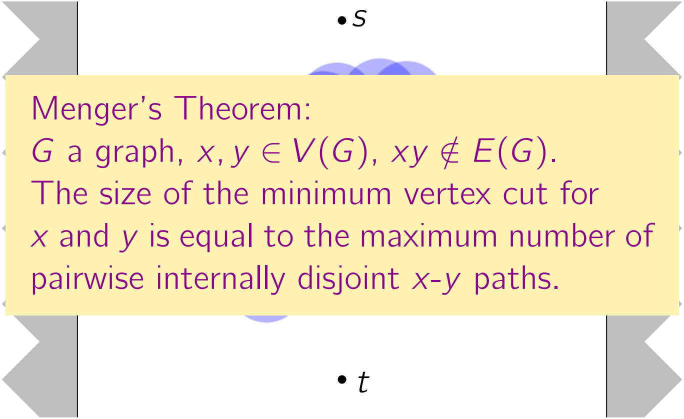

Menger's Theorem:
G a graph, $x, y \in V(G)$, $xy \notin E(G)$.
The size of the minimum vertex cut for x and y is equal to the maximum number of pairwise internally disjoint x-y paths.

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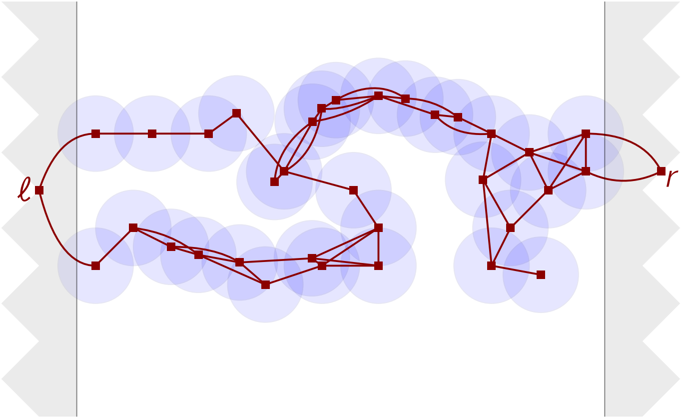

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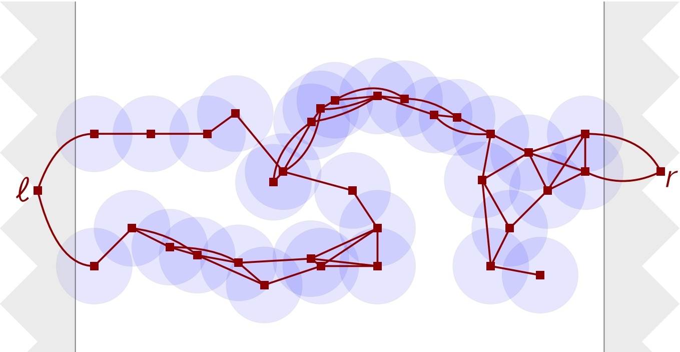

$$
\begin{aligned}
\text{OPT} &\geq \text{max nb } \ell\text{-}r \text{ vertex-disjoint paths}\\
&= \text{min size } \ell\text{-}r \text{ vertex cut}\\
&\geq \text{OPT}
\end{aligned}
$$

Barrier resilience - Rectangular domain in P

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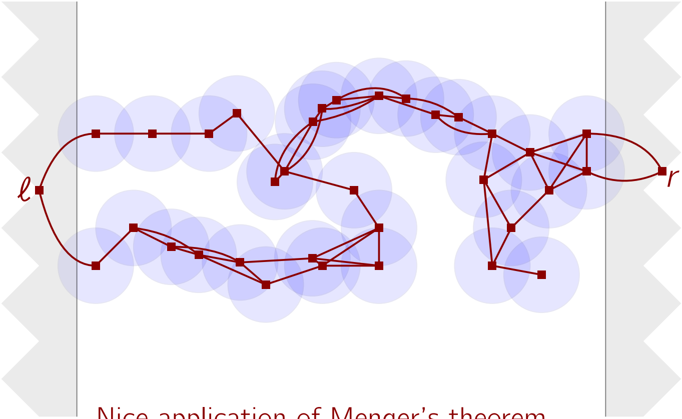

Nice application of Menger's theorem
No Menger-like theorem for the annular case

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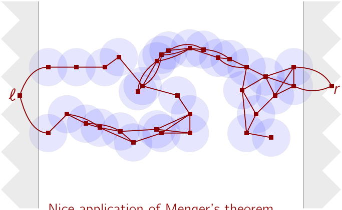

Nice application of Menger's theorem
No Menger-like theorem for the annular case

- finding the minimum ℓ-r cut is a max-flow problem
- one can use geometric data structures to solve it in $O(n^{3/2} \operatorname{polylog} n)$ time [C., Mulzer 2021]
- can it be solved in near-linear time?

Related problems

If you cannot solve a problem, perturb it:

Related problems

If you cannot solve a problem, perturb it:

- other shapes instead of disks
  - the rectangular case can be solved in polynomial time for any reasonable shape; same argument
  - the annular case is NP-hard for (unit) segments or crossing rectangles

[Alt, C., Giannopoulos, Knauer 2017]
[Korman, Löffler, Silveira, Strash 2018]
[Tseng and Kirkpatrick 2011]

- the annular case is FPT wrt OPT for any connected shape

[Eiben and Lokshtanov 2020]

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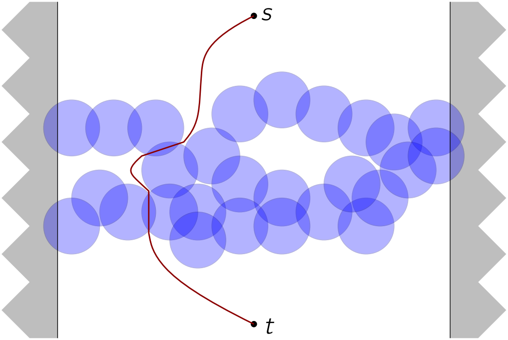
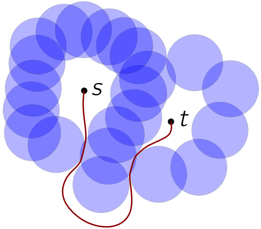
- the annular case is FPT wrt OPT for any connected shape
  [Eiben and Lokshtanov 2020]
- change the criteria
  - shrink the disks, instead of deleting them

Minimum shrinking problem

min $\sum$ shrinking [C., Jain, Lubiw, Mondal 2018]



rectangle



annulus

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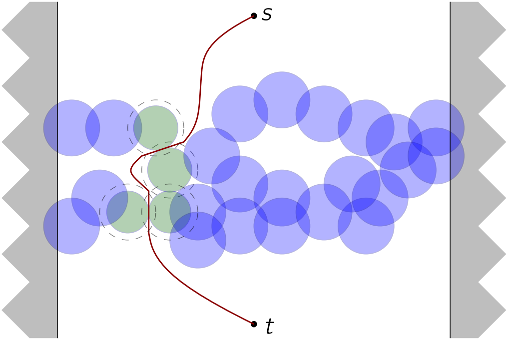

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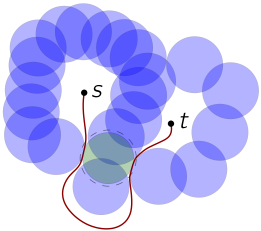

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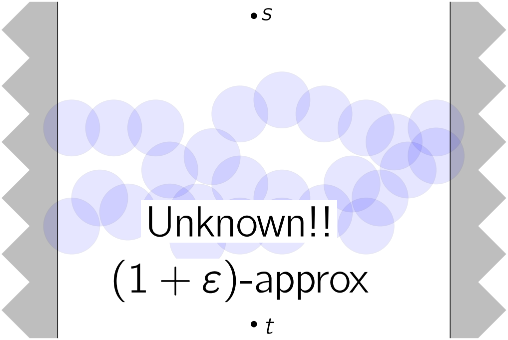

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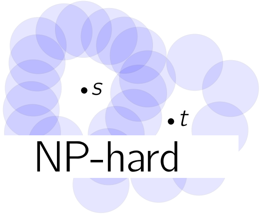
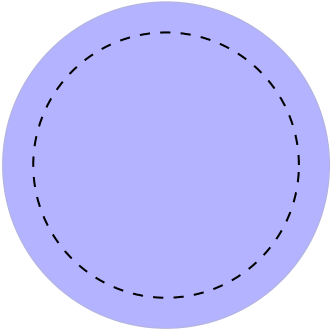


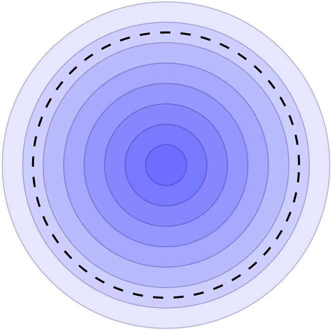
annulus

Minimum shrinking problem - Approximation

Key ideas: [C., Jain, Lubiw, Mondal 2018]

- get some poly-approximation
- discretize the shrinking
- connection to so-called minimum activation problems

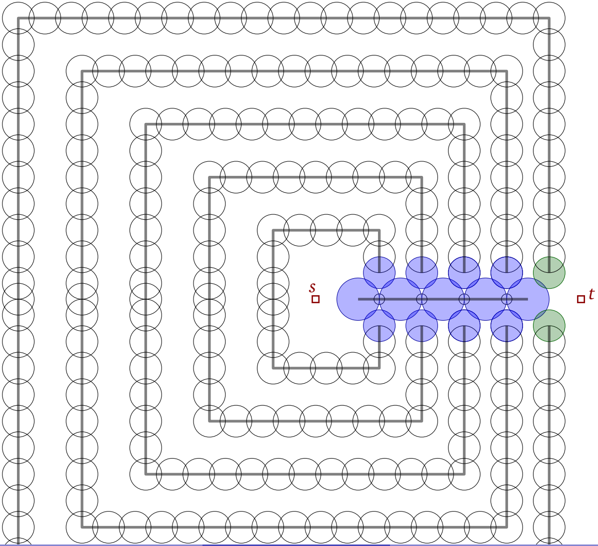




stack of concentric disks

Minimum shrinking problem - Hardness

[C., Colin de Verdière 2020]



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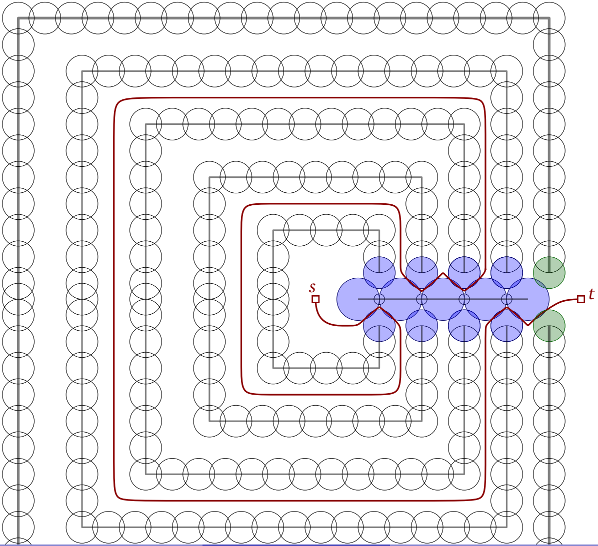

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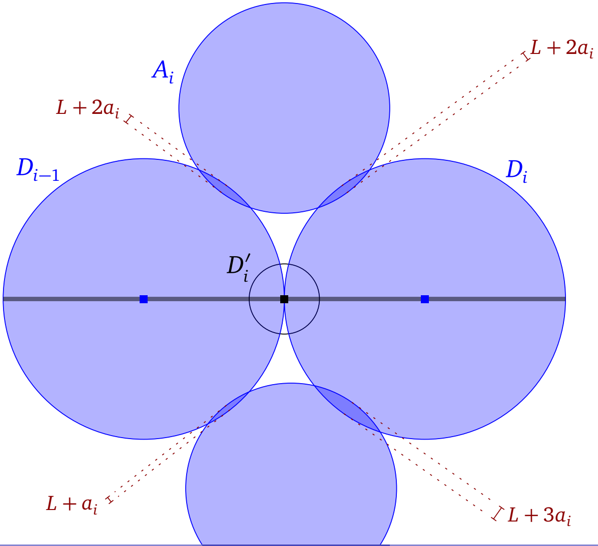


State of the art

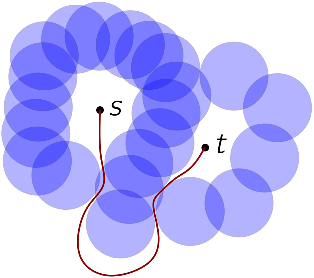
	rectangular domain	annular domain
barrier problem total failure	polynomial, $\tilde{O}(n^{3/2})$ Menger's theorem max flow	unknown complexity FPT PTAS in some cases
shrinking barrier min $\sum$ shrinking	unknown complexity $(1+\varepsilon)$-approx in $O(n^5/\varepsilon^{2.5})$ time	(weakly!!) NP-hard different radii

unit disks vs. disks vs. pseudodisks

Outline

Two independent parts

- Barrier resilience
- Crossing numbers of graphs



barrier

A short interlude

How to start a paper; two examples

A short interlude

How to start a paper; two examples

The Magical Number Seven, Plus or Minus Two...
George A. Miller (1956), Harvard University
Psychological Review, 63, 81-97

My problem is that I have been persecuted by an integer. For seven years this number has followed me around, has intruded in my most private data, and has assaulted me from the pages of our most public journals. This number assumes a variety of disguises, being sometimes a little larger and sometimes a little smaller than usual, but never changing so much as to be unrecognizable...

A short interlude

How to start a paper; two examples

On Hodge-Riemann Cohomology Classes
Julius Ross and Matei Toma
arXiv 2106.11285

Since the dawn of time, human beings have asked some fundamental questions: who are we? why are we here? is there life after death? Unable to answer any of these, in this paper we will consider cohomology classes on a compact projective manifold that have a property analogous to the Hard-Lefschetz Theorem and Hodge-Riemann bilinear relations.

Outline

Two independent parts

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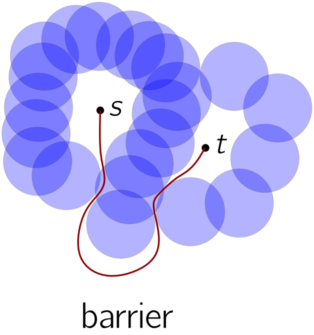


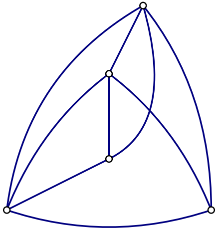
Drawing of a graph

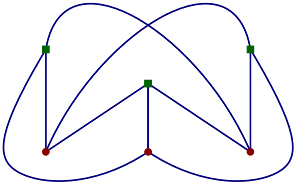
Drawing D of a graph G (in the plane)

- each vertex one point - injectively
- each edge one continuous, simple curve
- endpoints of edge uv are points for u and v
- the interior of an edge does not contain other vertices
- no common point in the interior of three edges

cr(D): number of crossings in drawing D

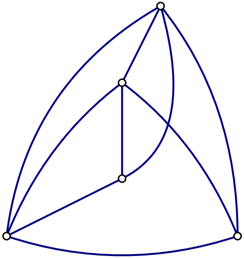
cr(G): mininum cr(D) over all drawings D of graph G

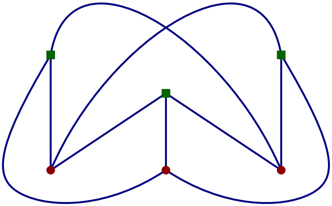




Crossing number 0

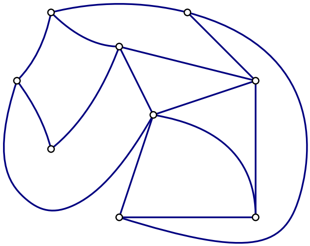
- $\mathrm{cr}(G) = 0$ if and only if G planar
- Good understanding of planar graphs
- Many algorithms specialized for planar graphs
- G planar $\iff$ G does not contain a subdivision of K_5 or $K_{3,3}$
- Efficient algorithms to recognize planar graphs

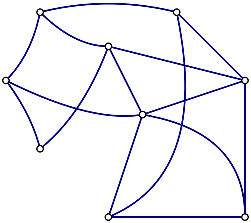




Why crossing number

Purchase, Cohen, James 1995: "Increasing the number of arc crossings in a graph decreases the understability of the graph".
Purchase, 1997: "reducing the number of edge crosses is by far the most important aesthetic".





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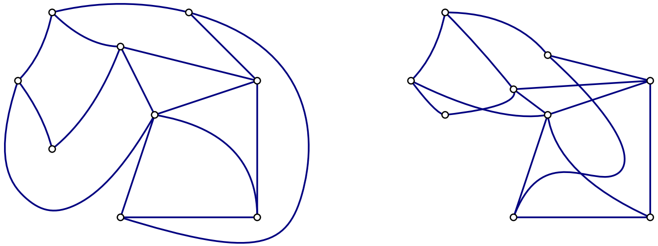
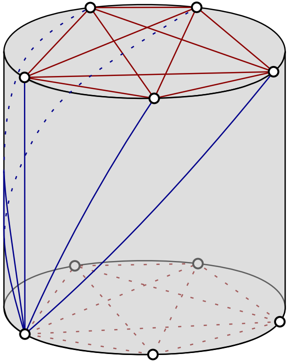


Huang, Eades, Hong, 2014: "The effect of crossing angles on human graph comprehension was validated."
Optimization, VLSI, Discrete Geometry, nice Mathematics and TCS

Classical conjectures – Harary-Hill

The crossing number of complete graphs is

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor \approx \frac{n^4}{64} \approx \frac{\binom{|E(K_n)|}{2}}{8}$$

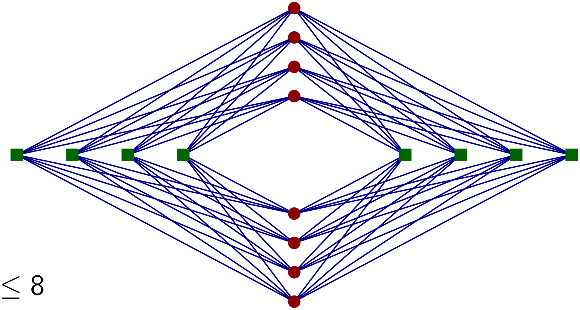


True for $n \leq 12$

Classical conjectures – Zarankiewicz-Turán

The crossing number of complete bipartite graphs (here for balanced)

$$\mathrm{cr}(K_{n,n}) = \left\lfloor \frac{n}{2} \right\rfloor^2 \left\lfloor \frac{n-1}{2} \right\rfloor^2 \approx \frac{n^4}{16} \approx \frac{\binom{|E(K_{n,n})|}{2}}{8}$$



True for $n \leq 8$

Computing crossing number

- NP-hard [Garey, Johnson 1983]
  [Several other proofs]
- APX-hard [C. 2013]
- Fixed parameter tractable wrt cr(G)
  quadratic time [Grohe 2005]
  linear time [Kawarabayashi, Reed 2007]
- NP-hard for bounded trewidth/pathwidth
  [Hlinený, Khazaliya 2024]
- Approximation: very complex and "bad" approximation factor
  State of the art [Chuzhoy, Tan 2022]
  - STOC 2022
  - degree bounded by Δ
  - $O\left(2^{O((\log n)^{7/8} \log\log n)} \cdot \mathrm{poly}(\Delta)\right)$-approximation
  - arXiv version has 379 pages

Planarity game

https://www.jasondavies.com/planarity/

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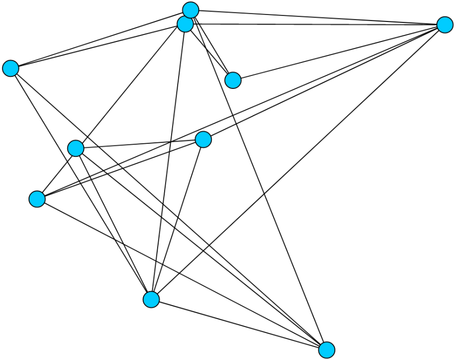
Planarity

Can you untangle the graph? See if you can position the vertices so that no two lines cross.

Number of line crossings detected: 36.



0 moves taken in 2.4s.

Number of vertices: 10 Generate new, random graph

Rectilinear drawings

Each edge drawn as a straight line segment

Rectilinear drawings

Each edge drawn as a straight line segment

$\overline{\mathrm{cr}}(G)\dots$ rectilinear crossing number of G

Rectilinear drawings

Each edge drawn as a straight line segment

$\overline{\mathrm{cr}}(G)\ldots$ rectilinear crossing number of G

- G planar $\iff \mathrm{cr}(G) = 0 \iff \overline{\mathrm{cr}}(G) = 0$

  [Wagner 1936, Fáry 1948]

Rectilinear drawings

Each edge drawn as a straight line segment

$\overline{\mathrm{cr}}(G)\ldots$ rectilinear crossing number of G

- G planar $\iff \mathrm{cr}(G) = 0 \iff \overline{\mathrm{cr}}(G) = 0$

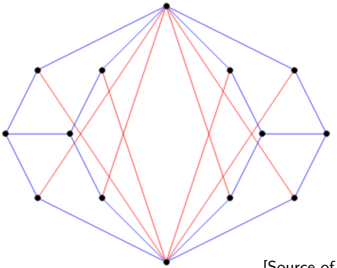
  [Wagner 1936, Fáry 1948]

- $\mathrm{cr}(G) = 1 \iff \overline{\mathrm{cr}}(G) = 1$ [Bienstock, Dean 1993]

  $\mathrm{cr}(G) = 2 \iff \overline{\mathrm{cr}}(G) = 2$

  Claims without proof that $\mathrm{cr}(G) = 3 \iff \overline{\mathrm{cr}}(G) = 3$

  For each $k \geq 4$ there is G with $\mathrm{cr}(G) = 4$ and $\overline{\mathrm{cr}}(G) = k$



[Source of the image: Stackexchange]

Many different crossing numbers

Different constraints for the drawings give different crossing numbers

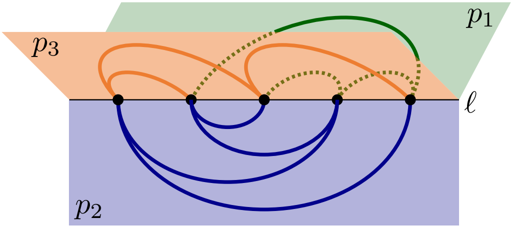
Huge amount of work

Survey by Marcus Schaefer has 166 pages

> **The Graph Crossing Number and its Variants: A Survey**
>
> Marcus Schaefer
> School of Computing
> DePaul University
> Chicago, Illinois 60604, U.S.A.
> `mschaefer@cdm.depaul.edu`
>
> Submitted: Dec 20, 2011; Accepted: Apr 4, 2013
> First Edition: Apr 17, 2013
> Second Edition: May 15, 2014
> Third Edition: Dec 22, 2017
> Fourth Edition: Feb 14, 2020
> Fifth Edition: Sep 4, 2020
> Sixth Edition: May 21, 2021
> Seventh Edition: Apr 8, 2022
> Mathematics Subject Classifications: 05C62, 68R10
>
> **Abstract**
>
> The crossing number is a popular tool in graph drawing and visualization, but there is not really just one crossing number; there is a large family of crossing number notions of which the crossing number is the best known. We survey the rich variety of crossing number variants that have been introduced in the literature for purposes

Example

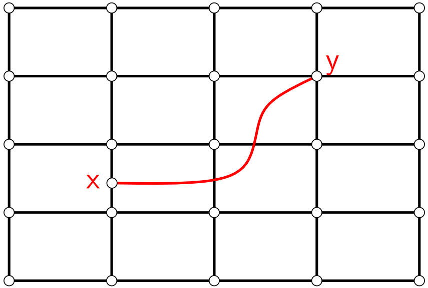
- Fixed order p-page book crossing number
  [Agrawal, C., Kaufmann, Saurabh, Sharma, Uno, Wolff 2024]
  - vertices fixed on a line ℓ; cannot be moved
  - pages (halfplanes) bounded by the line
  - each edge drawn in a single page



- And several related variants with editions

Near-planar graphs

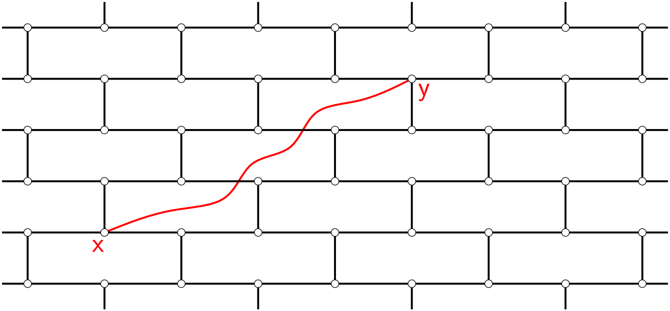
Non-planar H is near-planar if $H = G + xy$ for planar G



- weak relaxation of planarity
- near-planar $\subsetneq$ toroidal, apex

Near-planar – Riskin

- G planar, 3-connected, and 3-regular [Riskin 1996]
  - $cr(G + xy)$ attained by the following drawing: draw G planarly (unique) and insert xy minimizing crossings



Near-planar graphs are hard

[C., Mohar 2013]

Theorem

Computing $cr(G)$ for near-planar graphs is NP-hard.

Near-planar graphs are hard

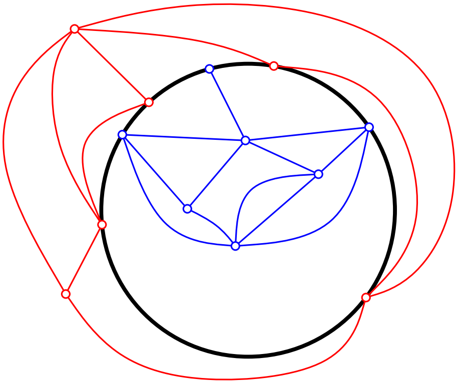
[C., Mohar 2013]

Theorem

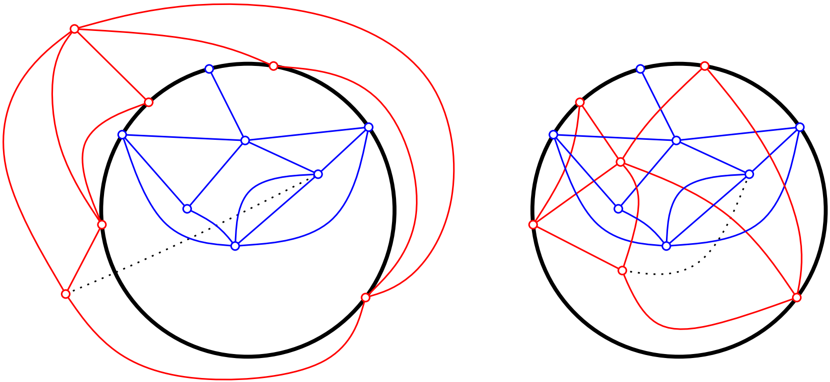
Computing $cr(G)$ for near-planar graphs is NP-hard.

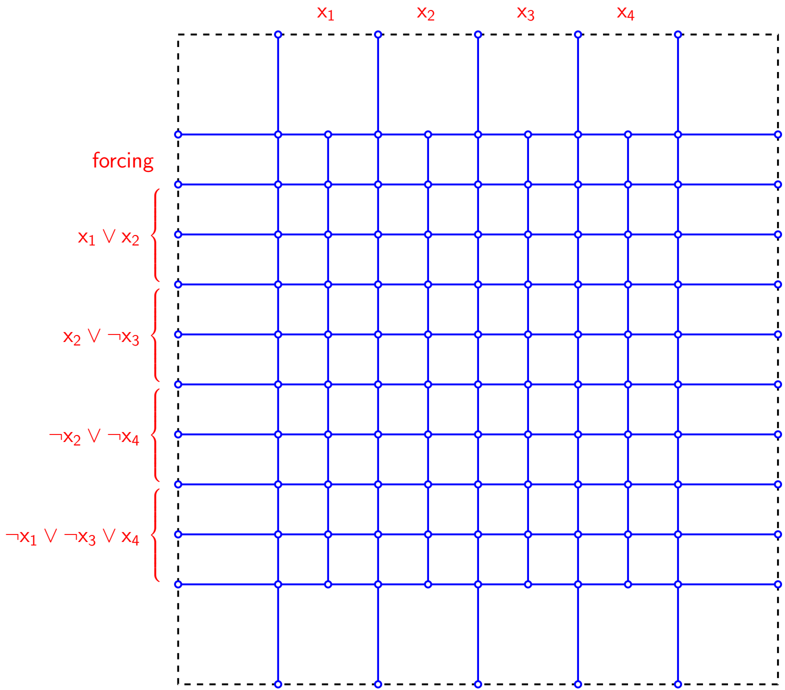
- adding one edge makes a big mess
- crossing number of toroidal graphs hard
- new reduction from SAT
  - previous reductions were from Linear Ordering
- new problem: red-blue anchored drawings

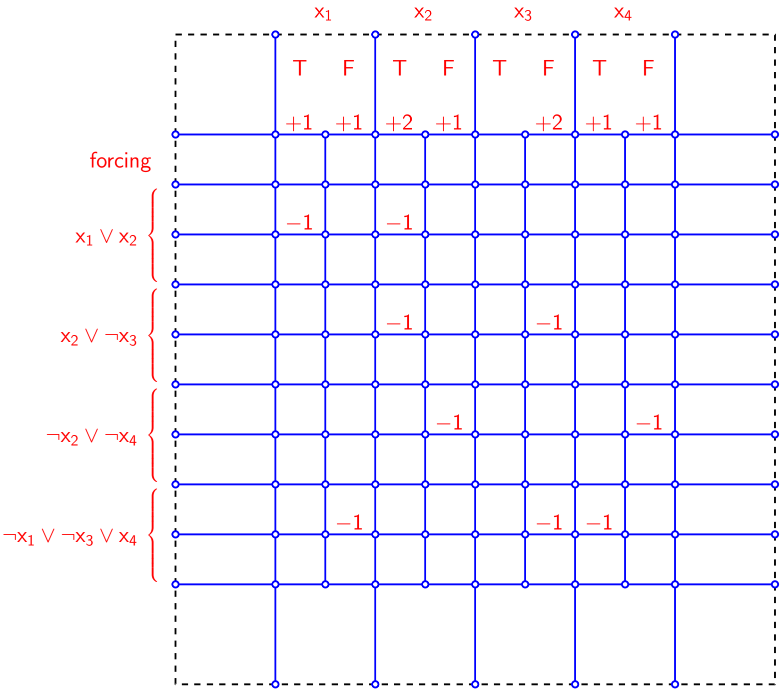
Red-blue anchored drawings?

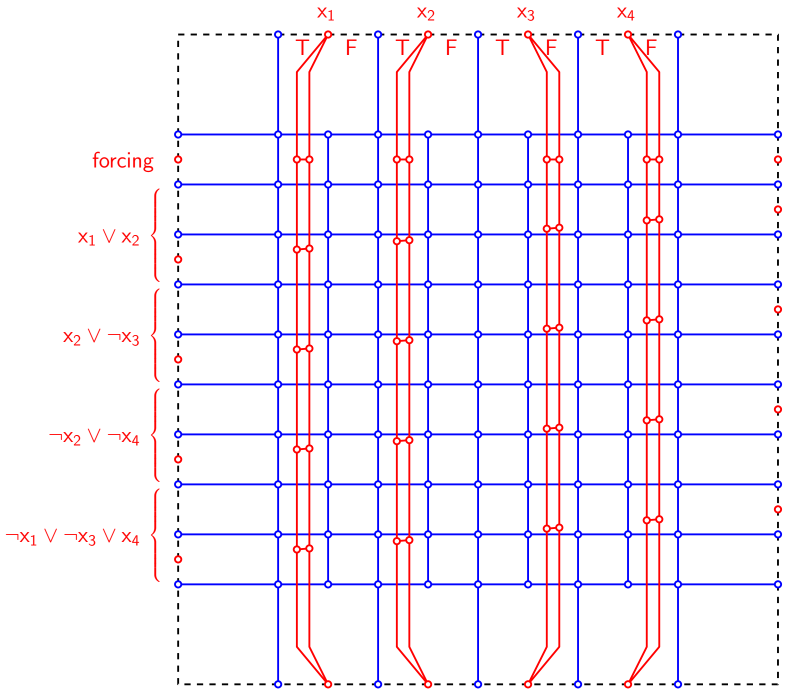


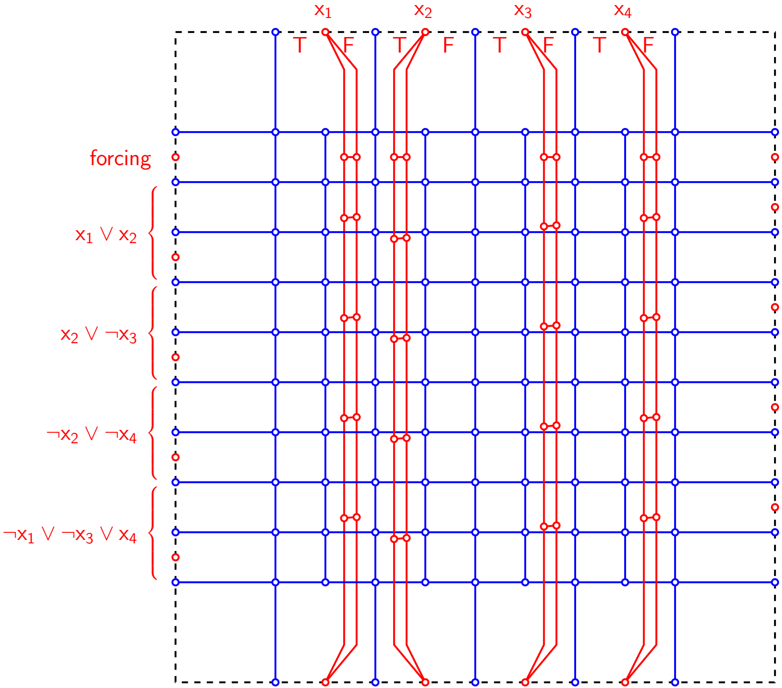
Red-blue anchored drawings?



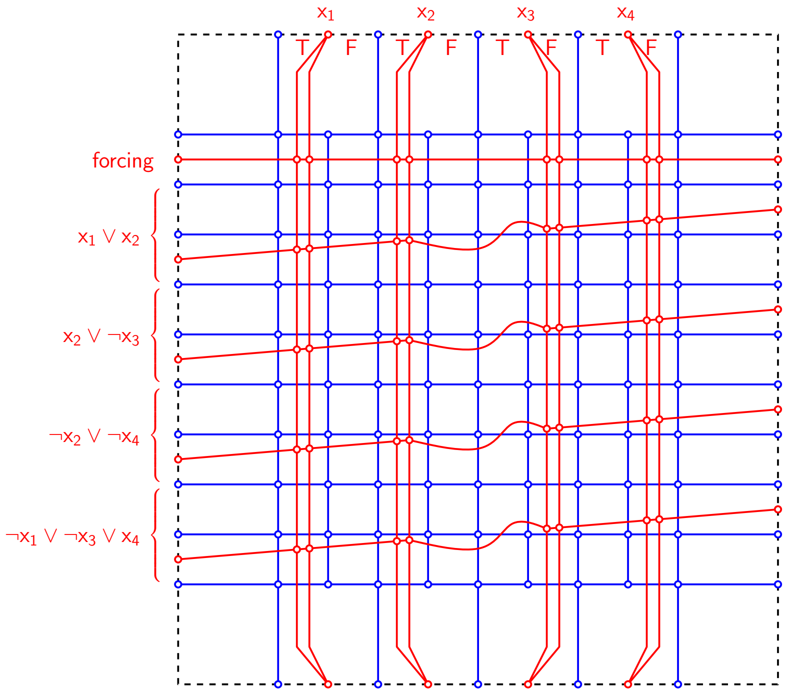

x_1
x_2
x_3
x_4
forcing
$x_1 \vee x_2$
$x_2 \vee \neg x_3$
$\neg x_2 \vee \neg x_4$
$\neg x_1 \vee \neg x_3 \vee x_4$

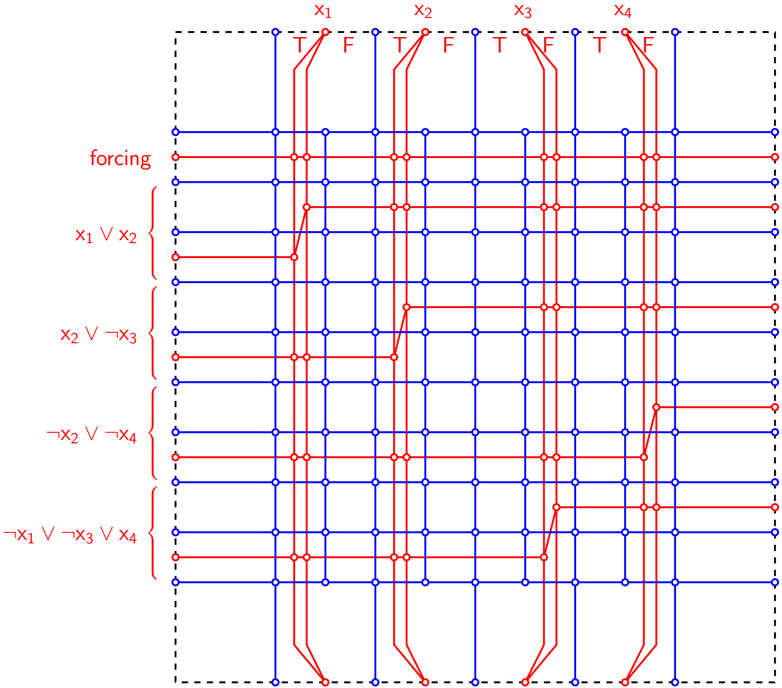

x_1
x_2
x_3
x_4
T
F
T
F
T
F
T
F
$+1$
$+1$
$+2$
$+1$
$+2$
$+1$
$+1$
forcing
$x_1 \vee x_2$
-1
-1
$x_2 \vee \neg x_3$
-1
-1
$\neg x_2 \vee \neg x_4$
-1
-1
$\neg x_1 \vee \neg x_3 \vee x_4$
-1
-1
-1


x_1 x_2 x_3 x_4
T F T F T F T F
forcing
$x_1 \vee x_2$
$x_2 \vee \neg x_3$
$\neg x_2 \vee \neg x_4$
$\neg x_1 \vee \neg x_3 \vee x_4$



x_1
x_2
x_3
x_4
T F T F T F T F
forcing
$x_1 \vee x_2$
$x_2 \vee \neg x_3$
$\neg x_2 \vee \neg x_4$
$\neg x_1 \vee \neg x_3 \vee x_4$


x1
x2
x3
x4
T
F
T
F
T
F
T
F
forcing
$x_1 \vee x_2$
$x_2 \vee \neg x_3$
$\neg x_2 \vee \neg x_4$
$\neg x_1 \vee \neg x_3 \vee x_4$

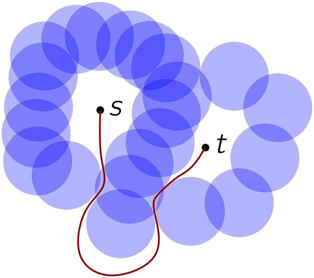

x1
x2
x3
x4
T
F
T
F
T
F
T
F
forcing
x1 ∨ x2
x2 ∨ ¬x3
¬x2 ∨ ¬x4
¬x1 ∨ ¬x3 ∨ x4

Crossing number of near-planar graphs

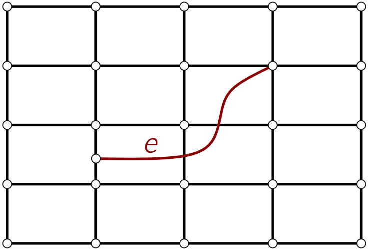
- adding one edge makes a big mess
- we need large degrees
  - three vertices of large degree suffice [Hliněný 2023]
- $\lfloor \Delta/2 \rfloor$-approximation [C., Mohar 2011]
  - number of edge-disjoint cycles separating x and y
  - number of vertex-disjoint cycles separating x and y
- Is it NP-hard for max degree 4?
- Research also on adding a vertex

- Similar proof for 1-planarity of near-planar graphs
  - each edge participates in at most one crossing
  - local crossing number

Conclusions



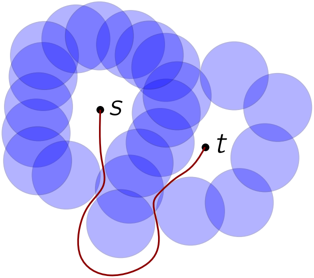
barrier



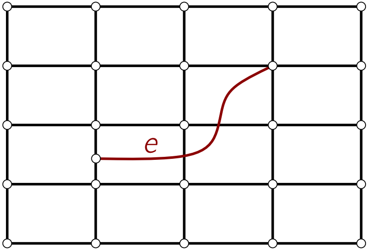
crossing number

near-planar

Conclusions



barrier



crossing number

near-planar

THANKS for your time!!