Disjoint Stable Matchings

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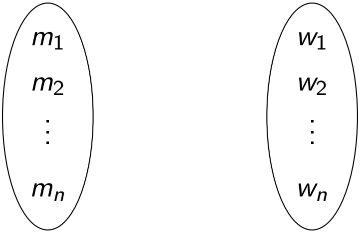
Matching in Bipartite Graphs

Bipartite Graph $G = (V, E)$



Set of men M   Set of women W

Matching

$M \subseteq E$ such that $\forall (m_i, w_j), (m_k, w_\ell) \in M \; i \neq k \iff j \neq \ell$ thus edges in a matching are disjoint.

Stable Matching Setup

Bipartite graph with preference ordering

For each man $m \in M$, there is a *preference ordering* π_m, which is a permutation over neighbors of m in G.
Similarly there is π_w for each $w \in W$, which is a permutation over neighbors of w in G.

Example for a complete graph:

$$\pi_{m_1} = \langle w_4, w_3, \cdots, w_n \rangle \qquad \pi_{w_1} = \langle m_1, m_3, \cdots, m_n \rangle$$

$$\pi_{m_2} = \langle w_1, w_6, \cdots, w_9 \rangle \qquad \pi_{w_2} = \langle m_5, m_2, \cdots, m_9 \rangle$$

$$\vdots \qquad \vdots$$

$$\pi_{m_n} = \langle w_5, w_7, \cdots, w_n \rangle \qquad \pi_{w_n} = \langle m_5, m_7, \cdots, m_n \rangle$$

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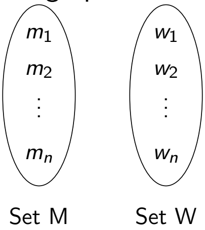

$$\pi_{w_1} = \langle m_1, m_3, \cdots, m_n \rangle$$
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$$\vdots$$
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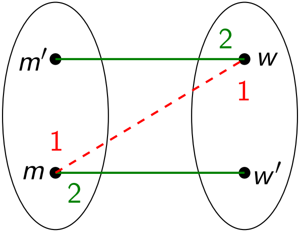
Blocking pair

A matching N is *blocked* by $(m, w) \notin N$ if $(m, w'), (m', w) \in N$ but $\pi_m(w) > \pi_m(w')$ *and* $\pi_w(m) > \pi_w(m')$.
Both m, w prefer each other more than their *partner* in N.



The Stable matching problem

Stable matching

A matching N is *stable* if it is *not* blocked by any pair (m, w).

This is the classical Stable Marriage Problem

M: set of men, W: set of women
A stable matching is a pairing of men and women so that no man-woman pair leaves their assigned partner and pairs up with each other.

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Motivation

- Practical: Models real-world situations like college admissions, assignment of medical interns to hospitals etc.
- Mathematical: Set of stable matchings has an interesting, rich structure.

Recognition (excerpt from Wikipedia)

Shapley and Roth were awarded the 2012 Nobel Memorial Prize in Economic Sciences "for the theory of stable allocations and the practice of market design". Gale had died in 2008, making him ineligible for the prize.

Example

$$M = \{m_1, m_2, m_3\},\ W = \{w_1, w_2, w_3\}$$

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Two stable matchings: red and green

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Purple matching is blocked by (m_2, w_3).

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Properties of stable matchings[1, 2, 3]

1. Existence: Every bipartite graph with preference ordering has at least one stable matching.
2. Computation: A stable matching can be efficiently computed by the Gale-Shapley algorithm.
3. Optimality: There is a unique stable matching that is simultaneously best for all $m \in M$, and simultaneously worst for all $w \in W$, (called M-optimal and W-pessimal stable matching). Similarly there is a W-optimal and M-pessimal stable matching.
4. Structure: Stable matchings in a bipartite graph form a distributive lattice under a suitably defined partial order. Conversely, for every finite distributive lattice L, there exists a stable matching instance whose lattice of stable matchings is isomorphic to L.

Stable Matchings

Stable Matching

A matching with no blocking pair

Checking stability: $O(n^2)$

Stable and Fixed Pairs

Stable Pair

A pair (m, w) is called as a *stable pair* if m and w are partners in at least one stable matching.

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Gale-Shapley Algorithm

Algorithm Gale-Shapley

```
procedure FIND STABLE MATCHING(M)
    assign each person to be free
    while some man m is free do
        w ← first woman on m's list to whom m hasn't proposed
        if w is free then
            assign m and w to be engaged to each other
        else
            if w prefers m to her current matched partner m′ then
                assign m and w to be engaged and m′ to be free
            else
                w rejects m                                    ▷ m remains free
            end if
        end if
    end while
    return Stable matching consisting of n engaged pairs
end procedure
```

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Men's Preference

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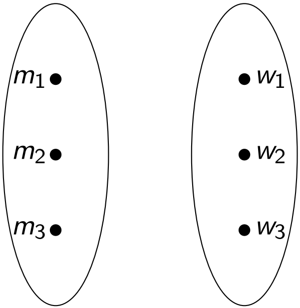
Women's Preference



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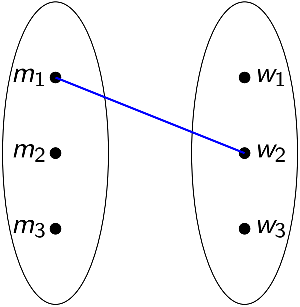

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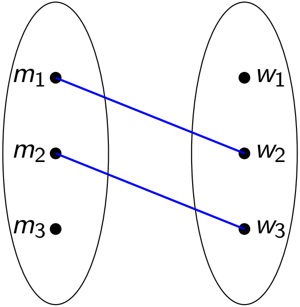

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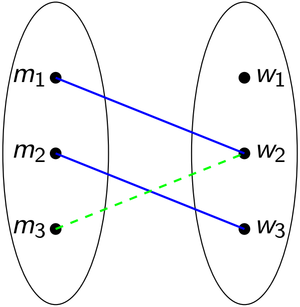

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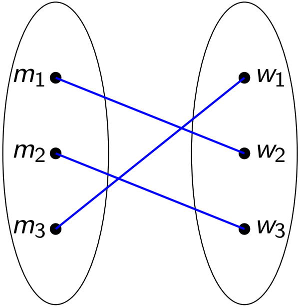

Gale-Shapley Algorithm - Key Results

1. Every stable matching instance has at least one stable matching.
2. All possible executions of the Gale-Shapley algorithm yield the same result.
3. If men propose, we get the *"Man-optimal"* stable matching.

   Man-optimal: Every man is matched with his best partner among all stable partners.
4. Reversing roles, i.e, women proposing, results in *"Woman-optimal"* stable matching.

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5. The man-optimal stable matching is *woman-pessimal*, and vice-versa.
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The partial order

A vertex $x \in M \cup W$ is said to *prefer* a matching P to a matching Q if $\exists y \neq z$ such that $(x, y) \in P, (x, z) \in Q$ and $\pi_x(y) > \pi_x(z)$.

Domination

A stable matching P is said to *dominate* a stable matching Q, written $P \preceq Q$, if for every $m \in M$, either m prefers P to Q or has the same partner in both P and Q.

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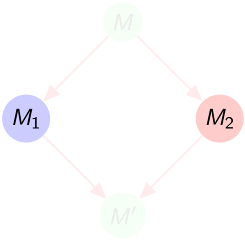

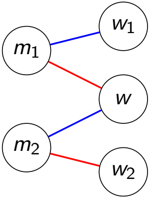

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M and M' are stable matchings!

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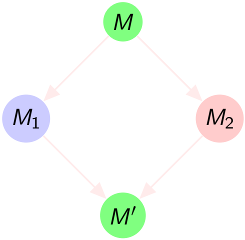

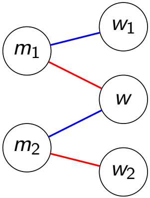

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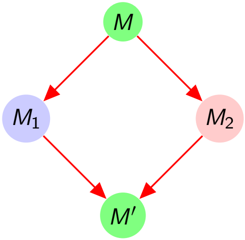

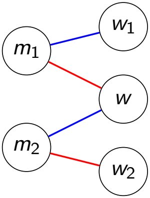

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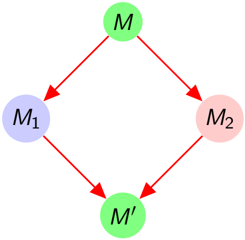

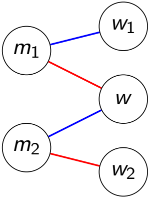

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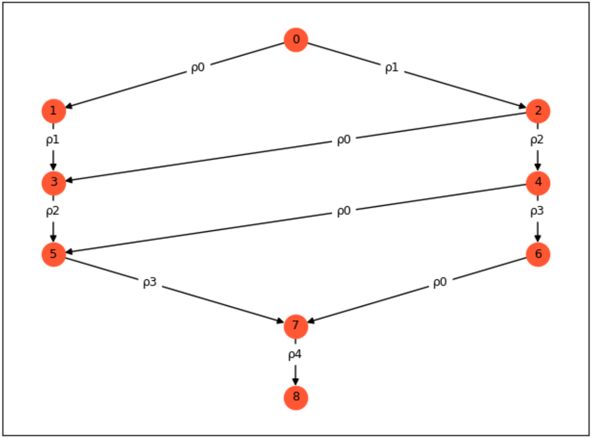
The Lattice Structure

Set of all stable matchings form a distributive lattice under the *Domination* relation.



Our problem and result

Our problem

Give an algorithm for a largest collection of *pairwise disjoint stable matchings*.

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A largest pairwise disjoint collection is a longest chain in the lattice. It can be found in linear time.

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A largest pairwise disjoint collection is a longest chain in the lattice. It can be found in linear time.

When does a non-trivial set of disjoint stable matchings exist?

Let P, Q be the M-optimal and the W-optimal stable matchings.
If $N \cap N' \neq \emptyset$, then every $(m, w) \in P \cap Q$ is in every stable matching.

This is because w is both the best stable partner and the worst stable partner of m.

Necessary condition

To have a non-trivial set of disjoint stable matchings, it is necessary that $P \cap Q = \emptyset$.

Proof idea

Theorem 1 (Teo-Sethuraman 1998[4])

Given a set S of stable matchigns, assigning k-th best partner to each $m \in M$ among all partners in S, gives a stable matching.

Let $P_1, P_2, \cdots, P_k$ be a largest set of disjoint stable matchings
Define $Q_i = \{(m, w) \mid w \text{ is } i\text{th best partner of m in } P_1, \ldots, P_k\}$.

Implication of Theorem 1

. $Q_1, \ldots, Q_k$ are all stable matchings, pairwise disjoint and form a chain.

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Proof idea continued

What do we get?

For any set of k pairwise-disjoint stable matchings, $\exists$ k-length chain of pairwise disjoint stable matchings.

Does it give us an algorithm?

Yes, just a modification of the Gale-Shapley algorithm!

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Disjoint Stable Matching Algorithm

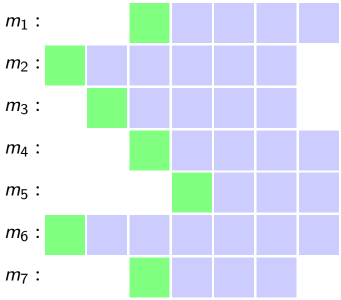
Step 1: Start with the men-proposing GS algorithm.

m_1 :							
m_2 :							
m_3 :							
m_4 :							
m_5 :							
m_6 :							
m_7 :							

Men's preference list

Disjoint Stable Matching Algorithm

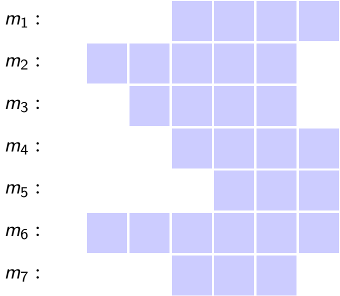
Step 2: Get the first stable matching M_1.



Men's Preference

Disjoint Stable Matching Algorithm

Step 3: Delete the matching edges. Continue GS algorithm.



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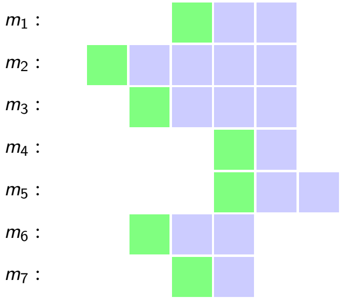
Disjoint Stable Matching Algorithm

Step 4: Get the next matching M_2. Delete it and repeat until the instance is empty.



Men's Preference

Termination and Time Complexity

In every iteration, we delete at least one entry from one preference list. As the total size of preference lists is $2n^2$, the algorithm **terminates**.

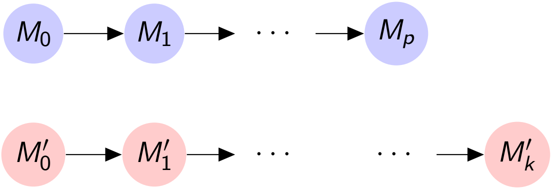
For the same reason, the running time of the algorithm is $\mathbf{O}(\mathbf{n^2})$.

Longest Chain of Disjoint Stable matchings

Correctness

The algorithm gives the longest chain of disjoint stable matchings.

Proof:



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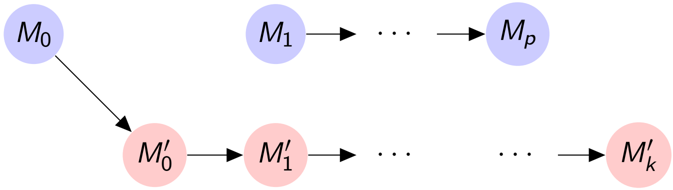

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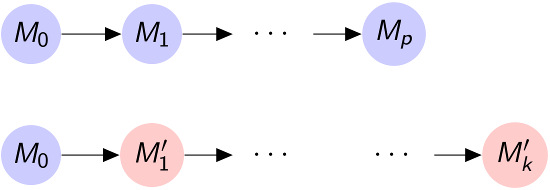

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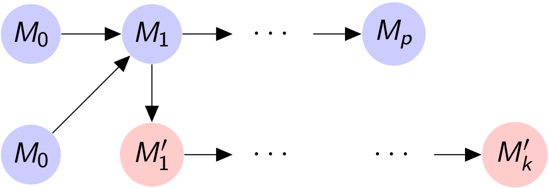

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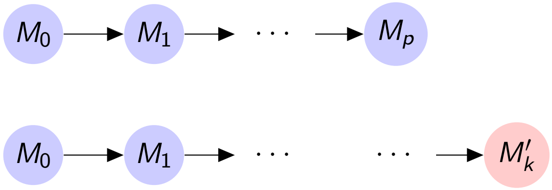

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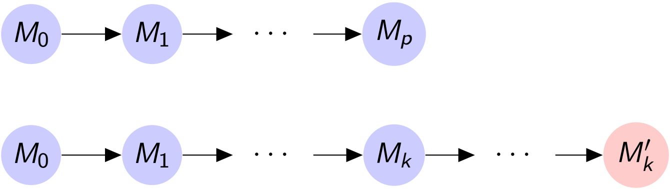

Random Instance

We analyze the number of maximum-length chains of disjoint stable matchings in a random stable matchings instance with complete lists.

Lemma 2

The probability of the number of maximum size chains of disjoint stable matchings exceeding $\left(\frac{n}{\ln n}\right)^{\ln n}$ *is at most* $O\left(\frac{(\ln n)^2}{n^2}\right)$.

Corollary 3

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Future Work

- Disjoint Stable Matchings in the Stable Roommates problem.
- When disjoint stable matchings do not exist, minimize pairwise intersection.

References

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