On Cayley graphs of monoids and semigroups

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Universität Oldenburg

Ernest Vidal
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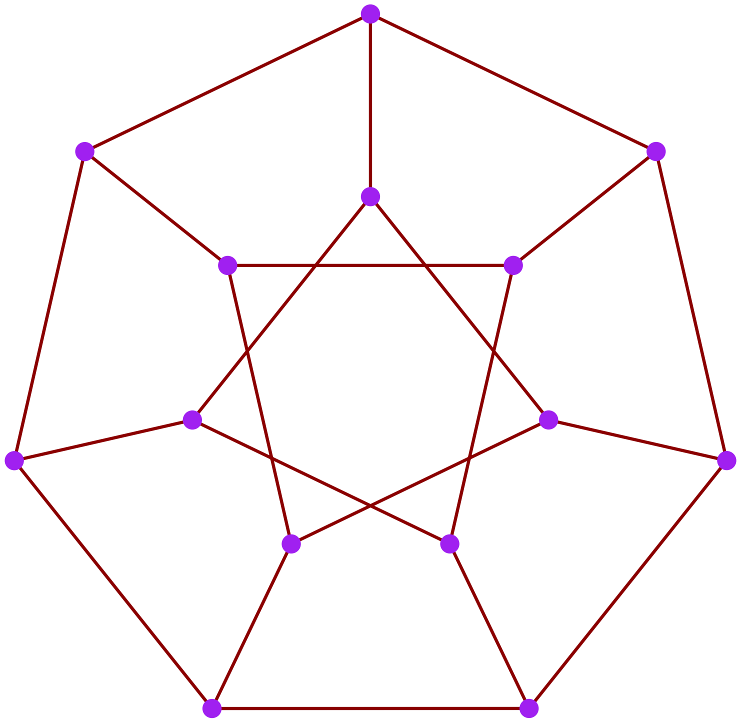
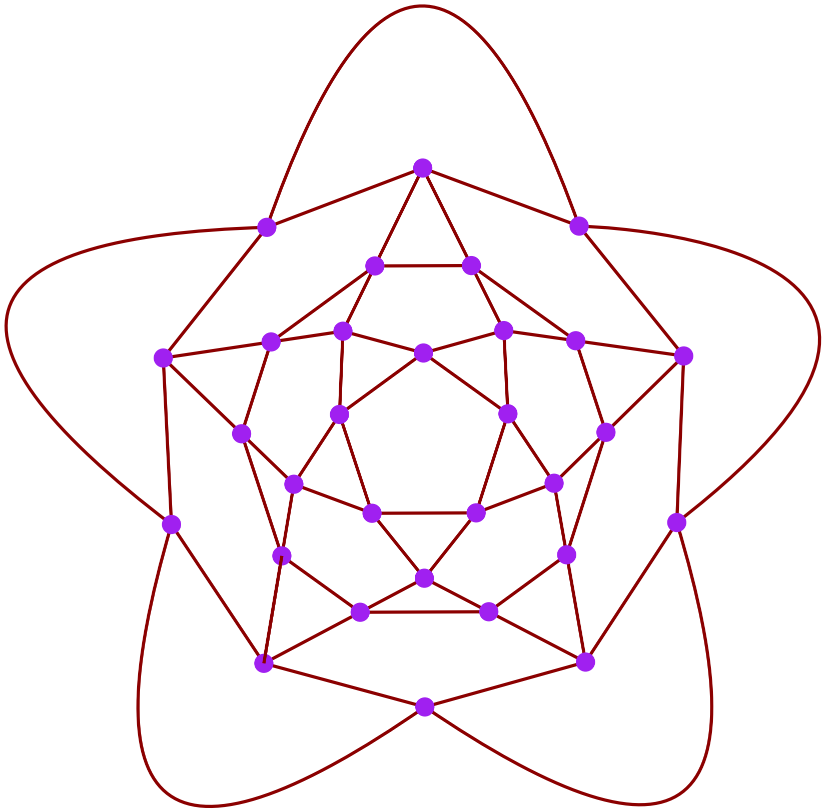
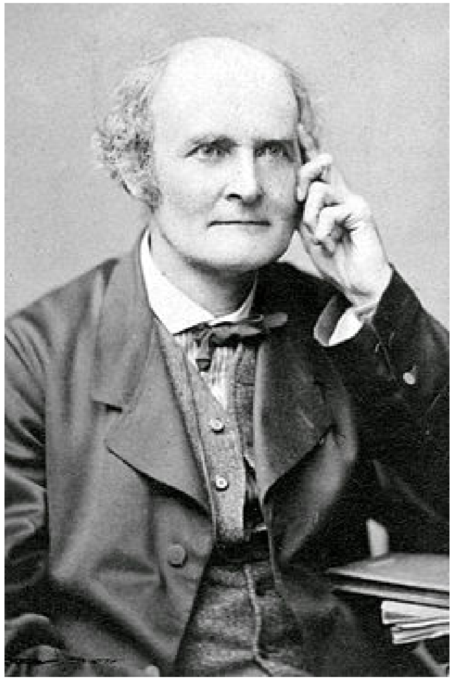




Cayley graphs

named after:
Arthur Cayley (1821-1895)



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Dehnsches Gruppenbild

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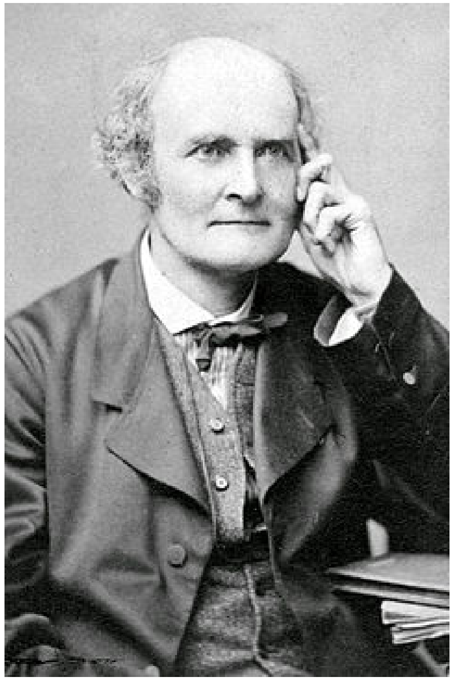
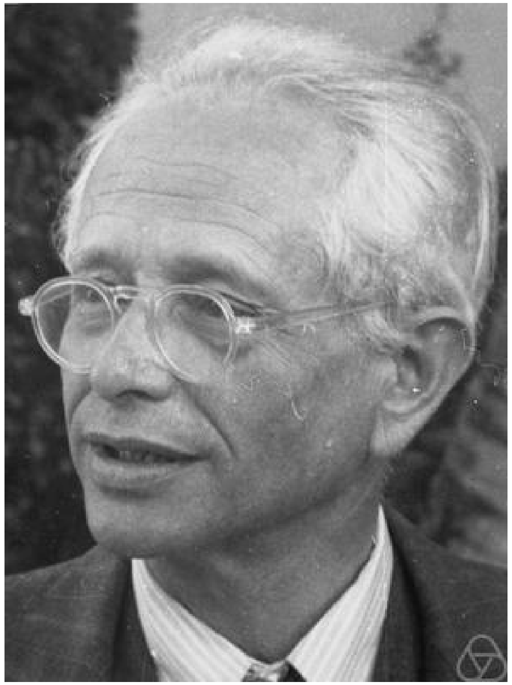


rediscovered by:
Max Dehn (1878-1952)



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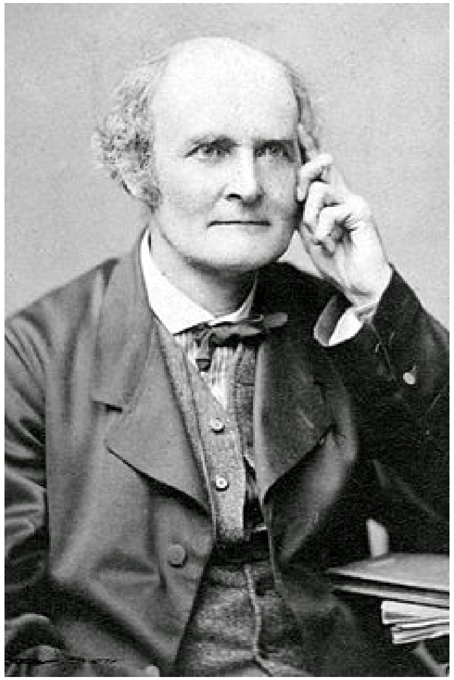
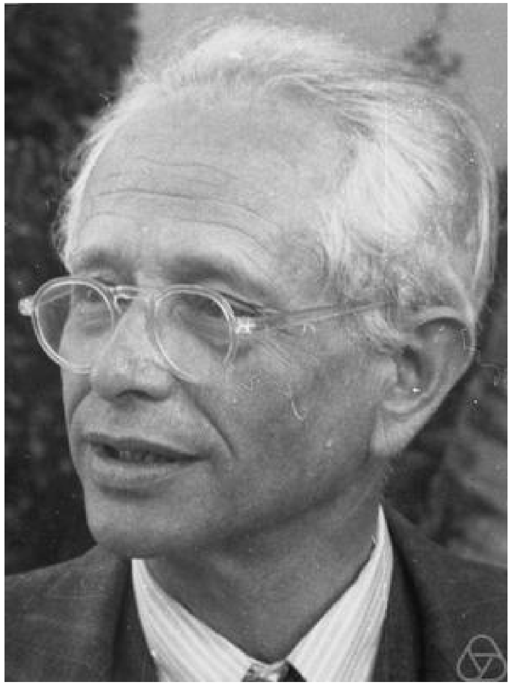


Cayley's paper published in 1878



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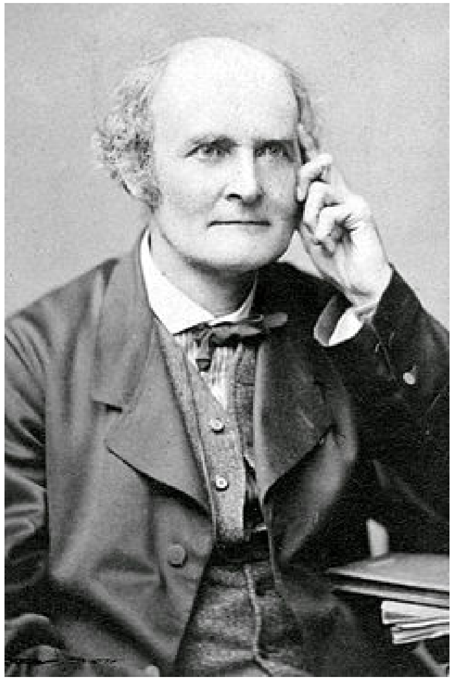

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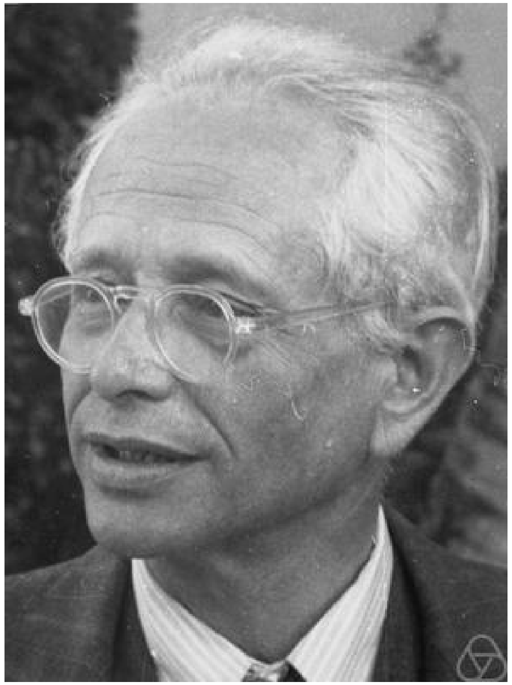

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(finite) set S with binary operation $\cdot : S \times S \to S$

associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in S$
neutral element: $\exists_{e \in S} : a \cdot e = e \cdot a = a$ for all $a \in S$
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S semigroup and $C \subseteq S \rightsquigarrow \mathrm{Cay}(S, C)$ (directed mutli) Cayley graph

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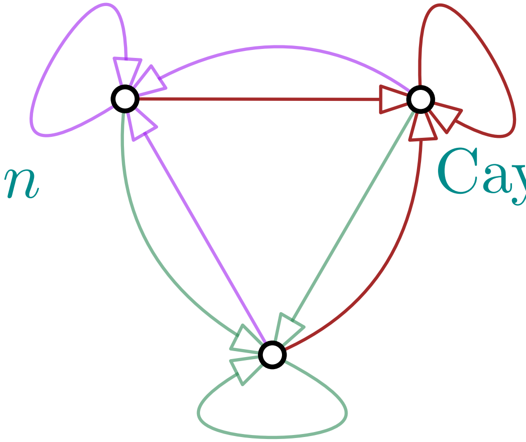

$\mathrm{Cay}(R_3, R_3)$

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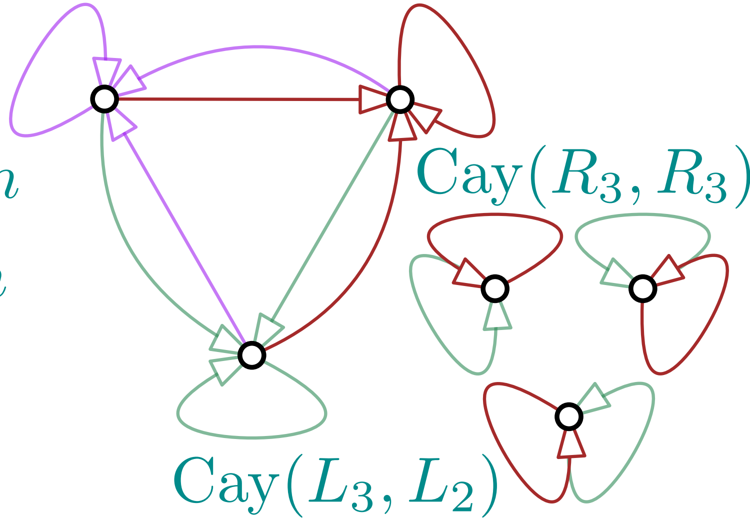

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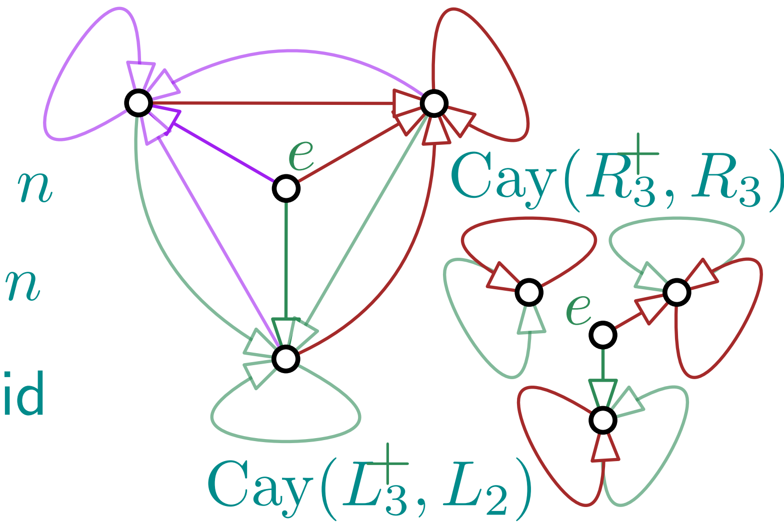

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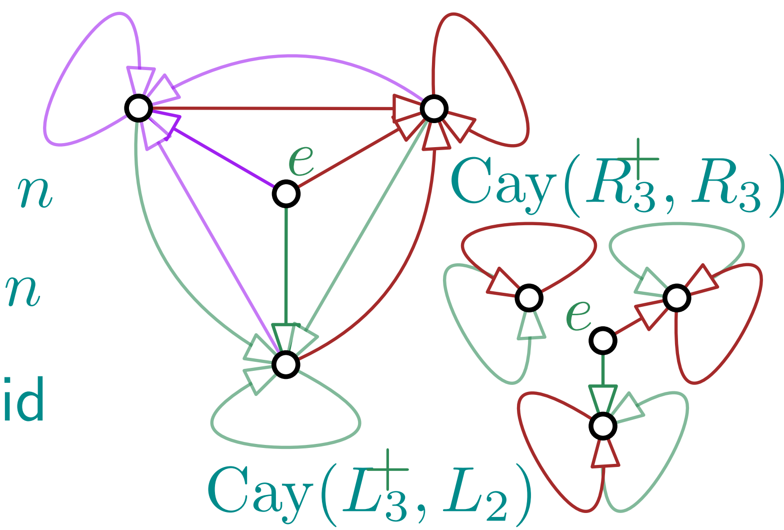
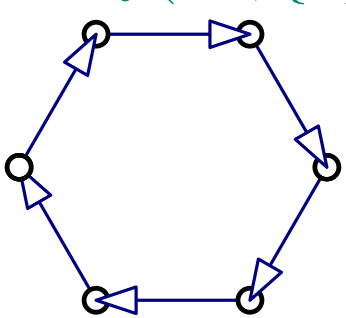


$\mathrm{Cay}(\mathbb{Z}_6, \{1\})$



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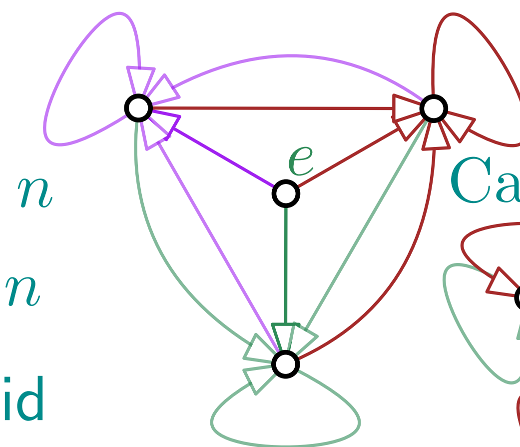

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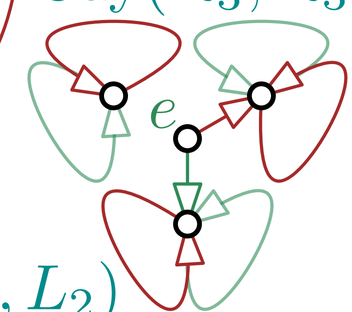

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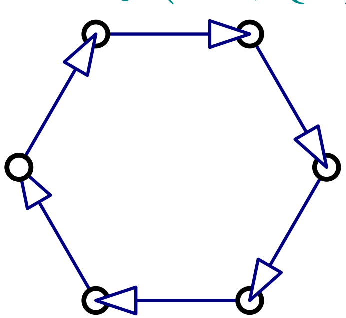

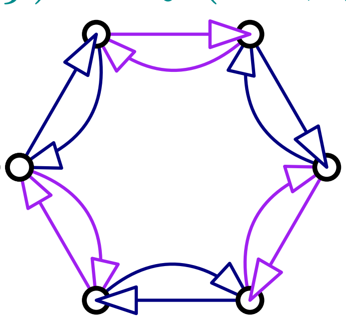

D_n=symmetry group of regular n-gon

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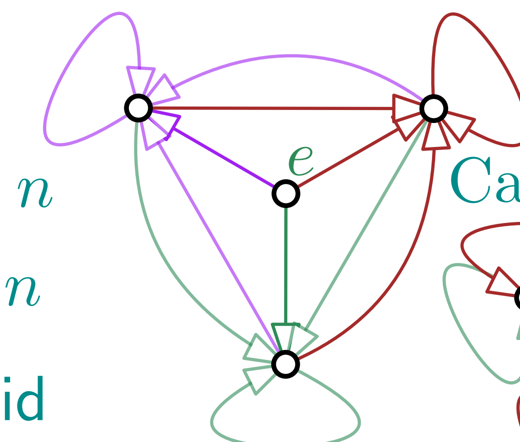

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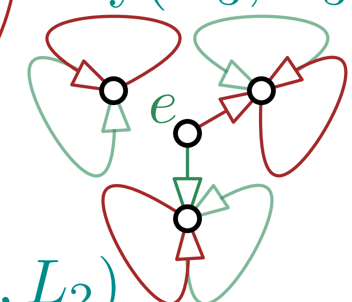

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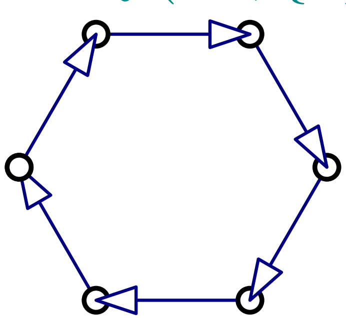

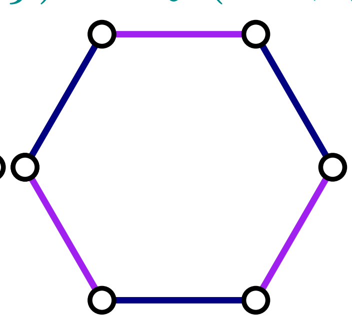

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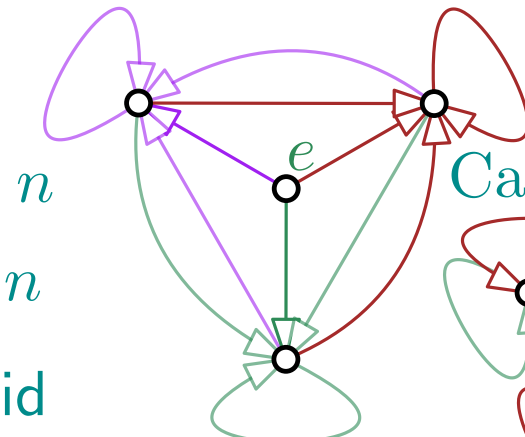

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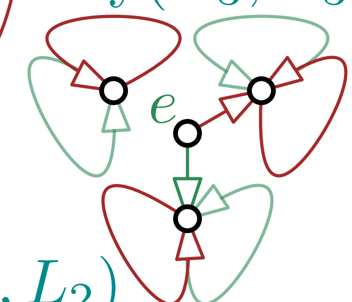

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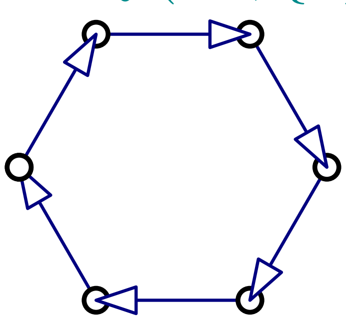

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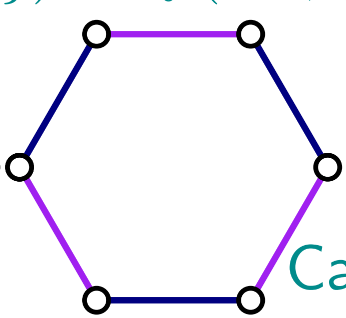

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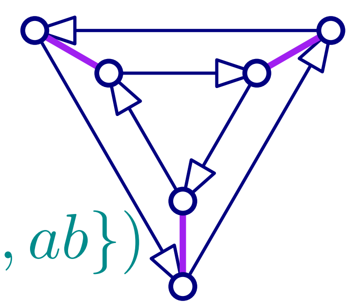

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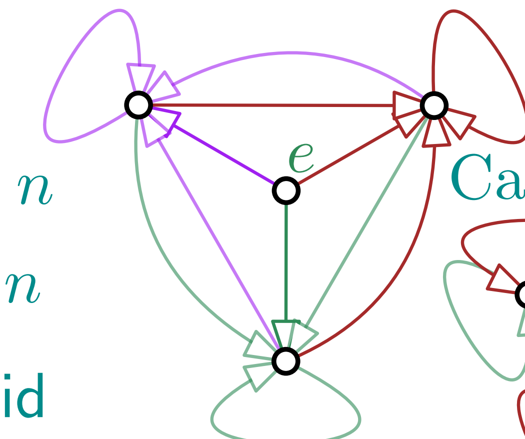

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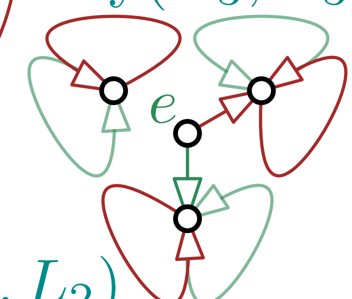

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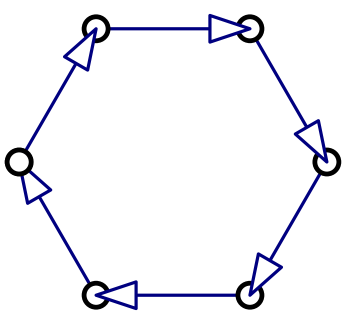

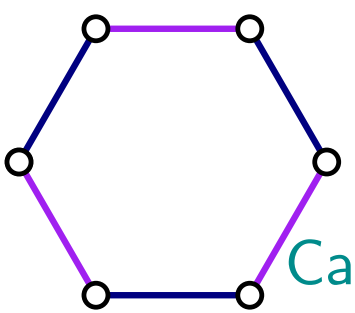

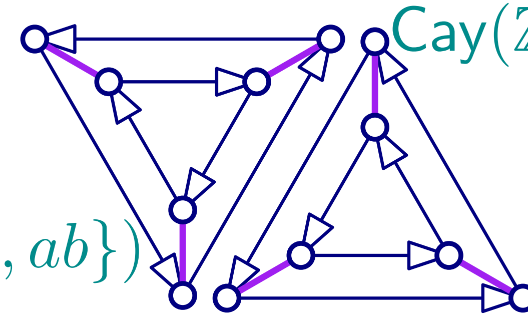

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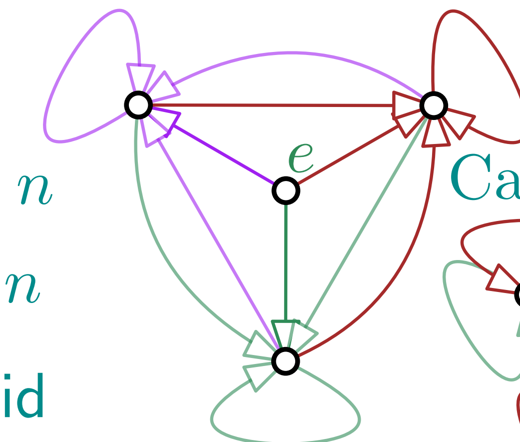

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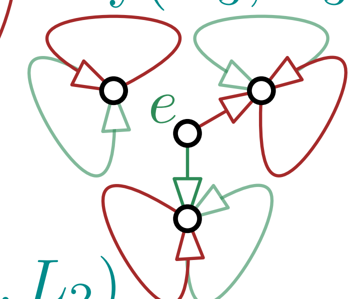

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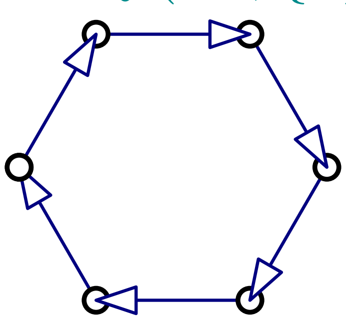
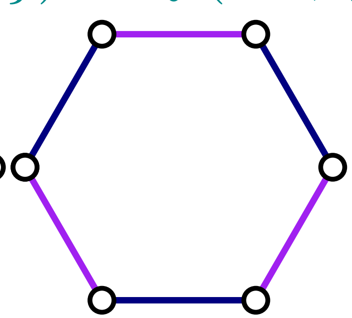




$\rightsquigarrow \mathrm{Cay}_{\mathrm{col}}(S, C)$ *colored* Cayley graph

color arcs with C

Cayley graphs

(finite) set S with binary operation $\cdot : S \times S \to S$

associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in S$ $\rightsquigarrow$ semigroup

neutral element: $\exists_{e \in S} : a \cdot e = e \cdot a = a$ for all $a \in S$ $\rightsquigarrow$ monoid

inverse element: $\forall_{a \in S} \exists_{a^{-1} \in S} : a \cdot a^{-1} = e$ $\rightsquigarrow$ group

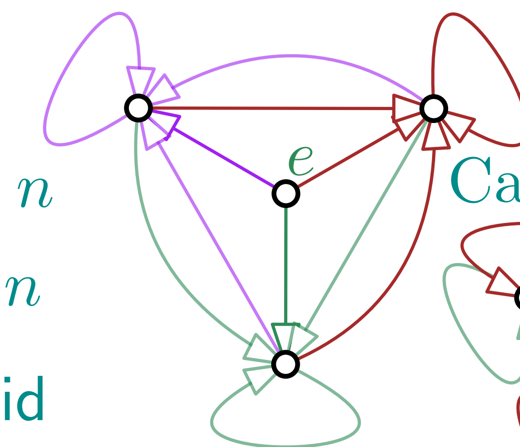
S semigroup and $C \subseteq S \rightsquigarrow \mathrm{Cay}(S, C)$ (directed mutli) Cayley graph

vertices $V = S$ *arcs* A: (s, t) for each $c \in C$ such that $s \cdot c = t$

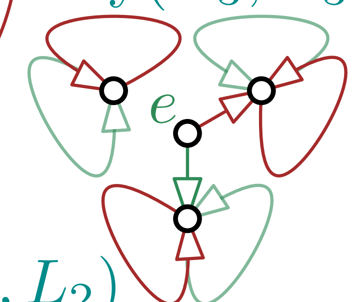
examples:

right-zero-band: $R_n : i \cdot j = j$ for all $1 \leq i, j \leq n$

left-zero-band: $L_n : i \cdot j = i$ for all $1 \leq i, j \leq n$

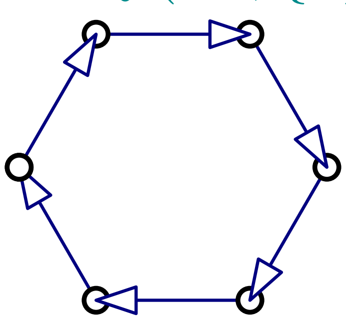


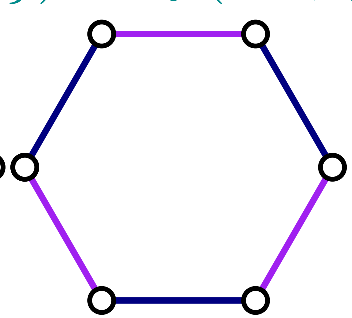
$\mathrm{Cay}(R_3^+, R_3)$



add neutral element: semigroup $S \to S^+$ monoid

$\mathrm{Cay}(\mathbb{Z}_6, \{1\})$ $\mathrm{Cay}(D_3, \{a, b\})$ $(a^2 = b^2 = (ab)^3 = e)$ $\mathrm{Cay}(L_3^+, L_2)$





$\rightsquigarrow \mathrm{Cay}_{\mathrm{col}}(S, C)$ *colored* Cayley graph

color arcs with C

$\rightsquigarrow \underline{\mathrm{Cay}}(S, C)$ (underlying) *simple* Cayley graph

forget directions, edge multiplicities, loops

Cayley graphs

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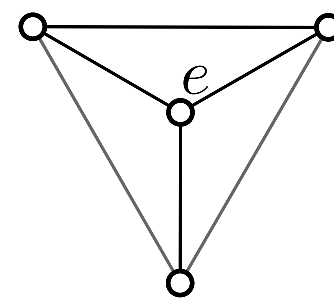

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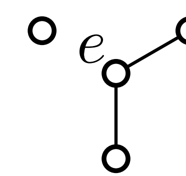

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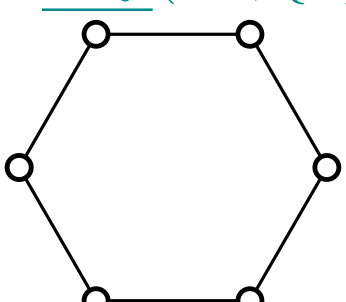

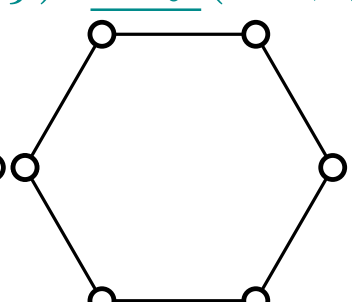

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color arcs with C

$\rightsquigarrow \mathrm{Cay}(S, C)$ (underlying) *simple* Cayley graph

forget directions, edge multiplicities, loops

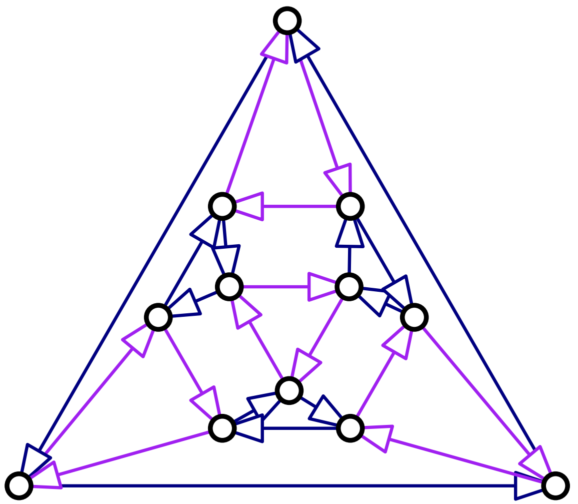
Planar groups

S *planar* if S has generating system C such that Cay(S, C) is planar

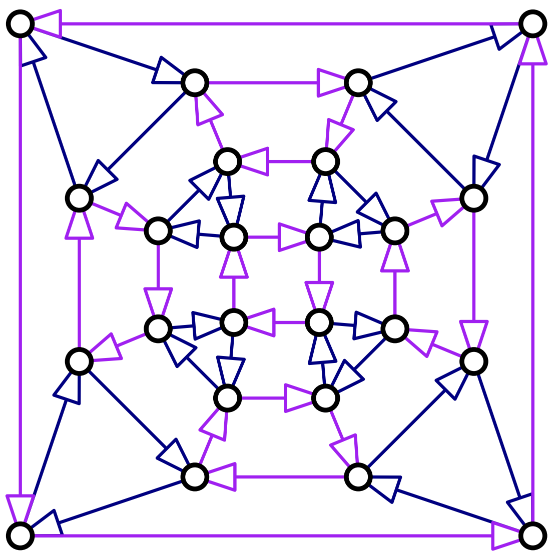
Thm [Maschke '96]: the planar groups are
$\mathbb{Z}_n$, D_n, A_4, S_4, A_5, $\mathbb{Z}_2 \times \mathbb{Z}_n$, $\mathbb{Z}_2 \times D_n$, $\mathbb{Z}_2 \times A_4$, $\mathbb{Z}_2 \times S_4$, $\mathbb{Z}_2 \times A_5$

Heinrich Maschke (1853-1908)

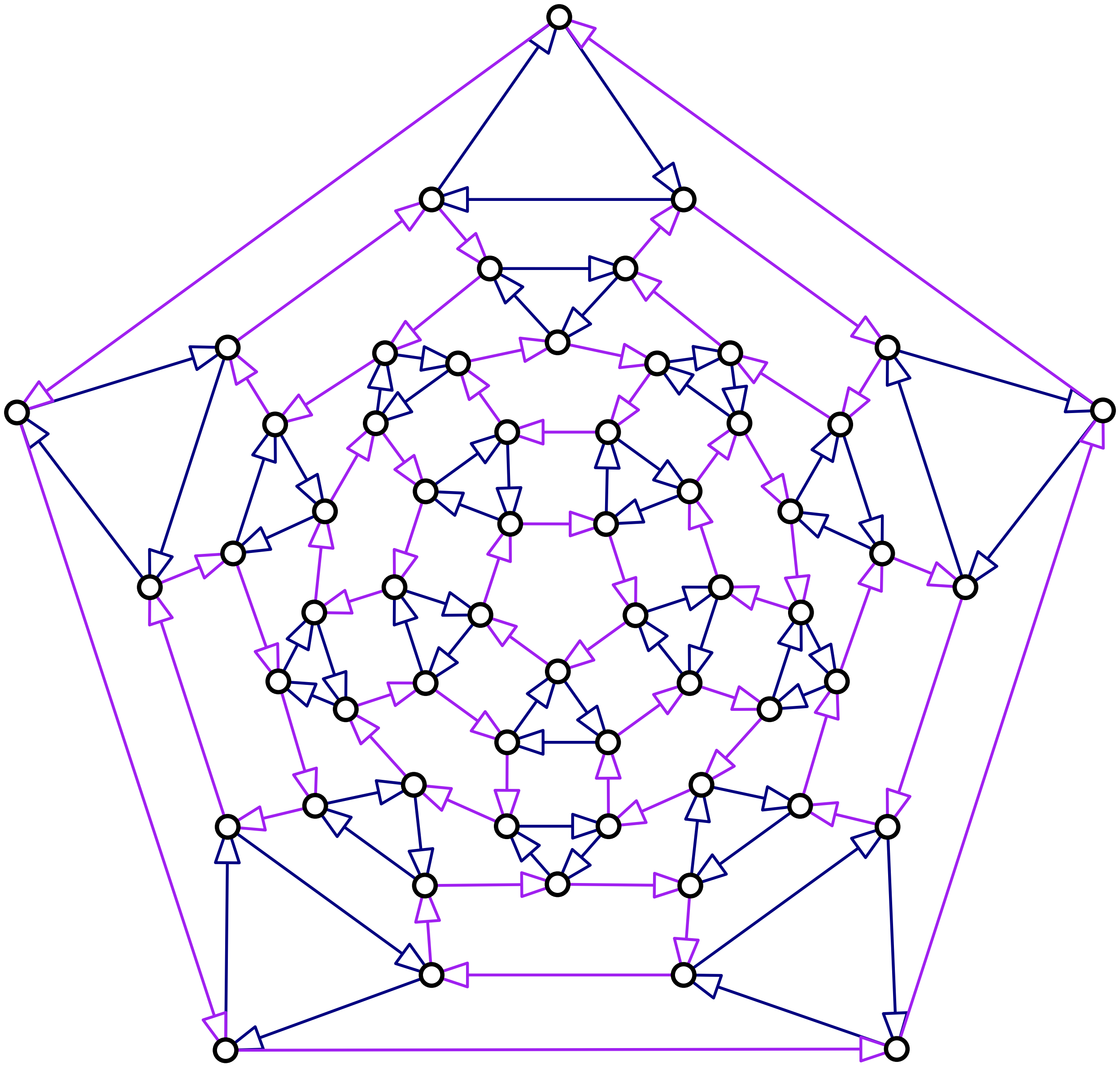
Case 1: two generators of order > 2



Cay$(A_4, \{(123), (234)\})$



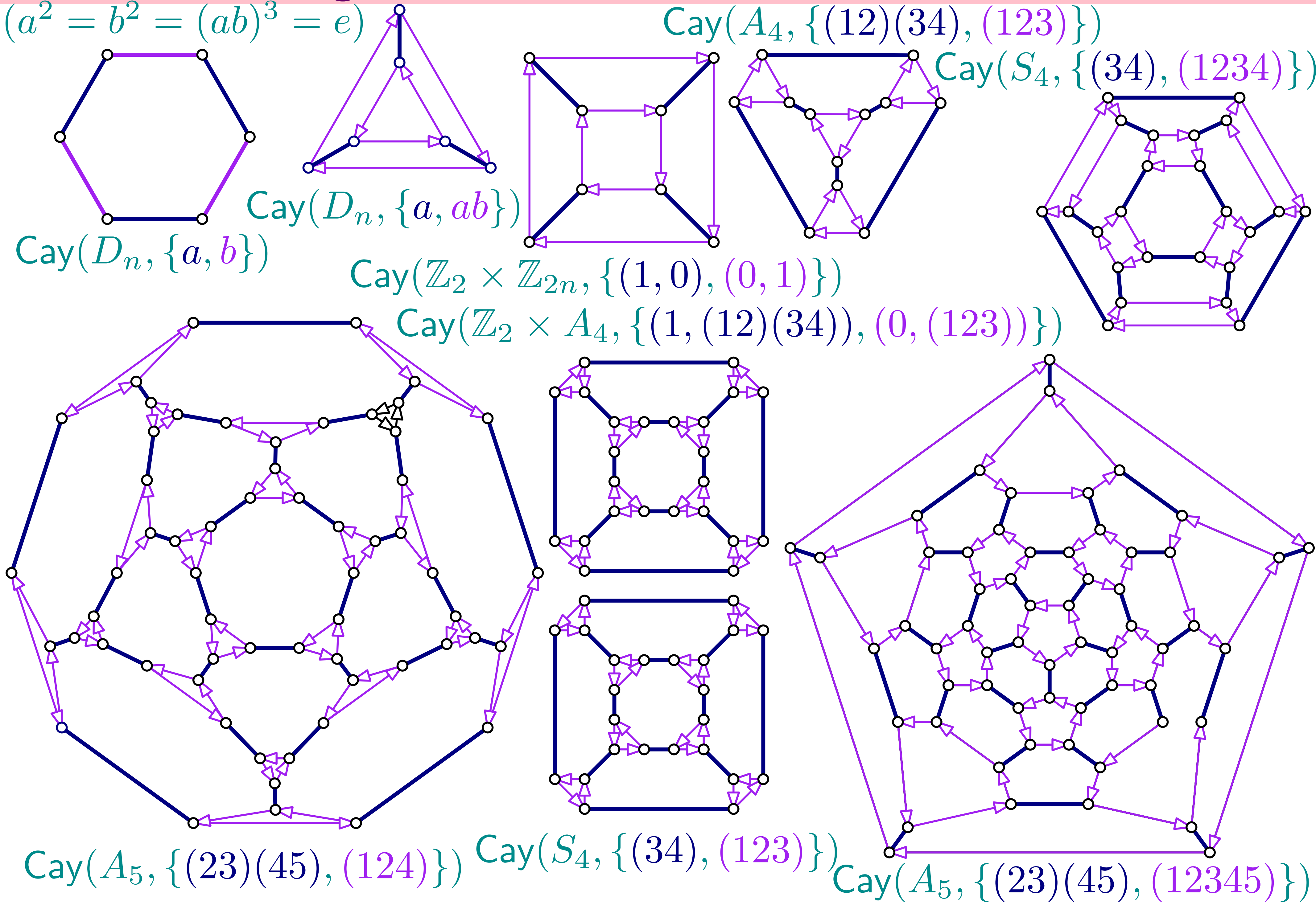
Cay$(S_4, \{(123), (1234)\})$



Cay$(A_5, \{(124), (12345)\})$

Case 2: generators of order 2 and > 2

$(a^2 = b^2 = (ab)^3 = e)$



$\mathsf{Cay}(D_n, \{a, b\})$

$\mathsf{Cay}(D_n, \{a, ab\})$

$\mathsf{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2n}, \{(1,0), (0,1)\})$

$\mathsf{Cay}(A_4, \{(12)(34), (123)\})$

$\mathsf{Cay}(S_4, \{(34), (1234)\})$

$\mathsf{Cay}(\mathbb{Z}_2 \times A_4, \{(1, (12)(34)), (0, (123))\})$

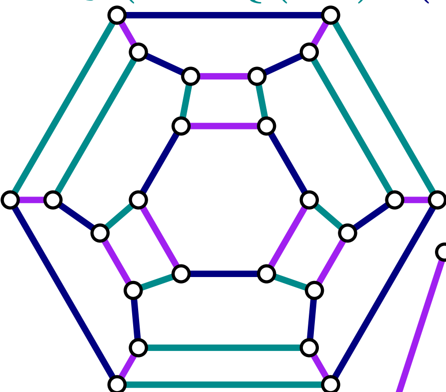
$\mathsf{Cay}(A_5, \{(23)(45), (124)\})$

$\mathsf{Cay}(S_4, \{(34), (123)\})$

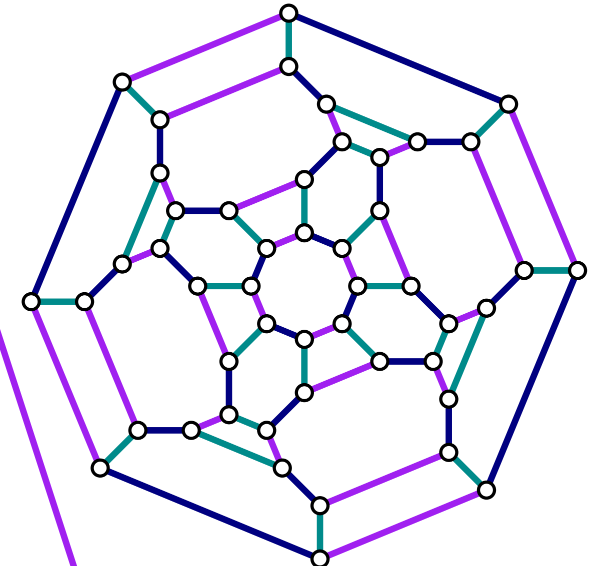
$\mathsf{Cay}(A_5, \{(23)(45), (12345)\})$

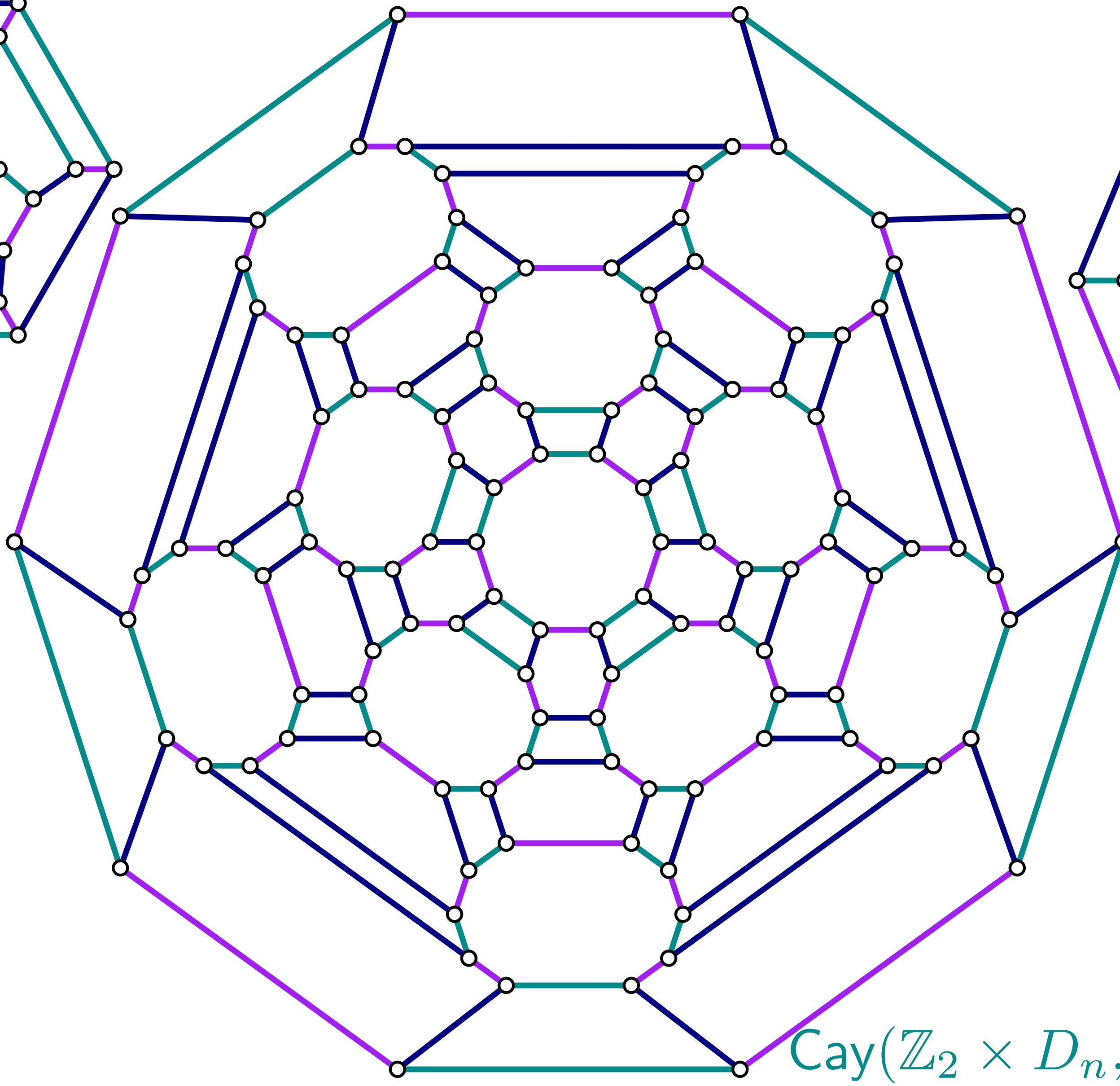
Case 3: three generators of order 2

$\mathsf{Cay}(S_4, \{(12), (23), (34)\})$

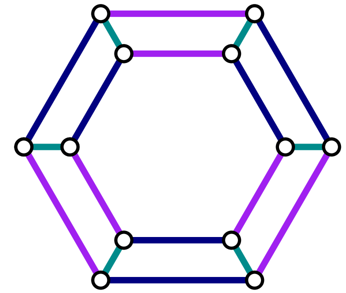


$\mathsf{Cay}(\mathbb{Z}_2 \times S_4, \{(0,(12)), (0,(23)), (1,(12)(34))\})$





$\mathsf{Cay}(\mathbb{Z}_2 \times A_5, \{(1,(12)(35)), (1,(24)(35)), (1,(23)(45))\})$



$\mathsf{Cay}(\mathbb{Z}_2 \times D_n, \{(1,e), (0,a), (0,b)\})$

Planar groups

Thm [Maschke '96]: the planar groups are
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Planar semigroups?

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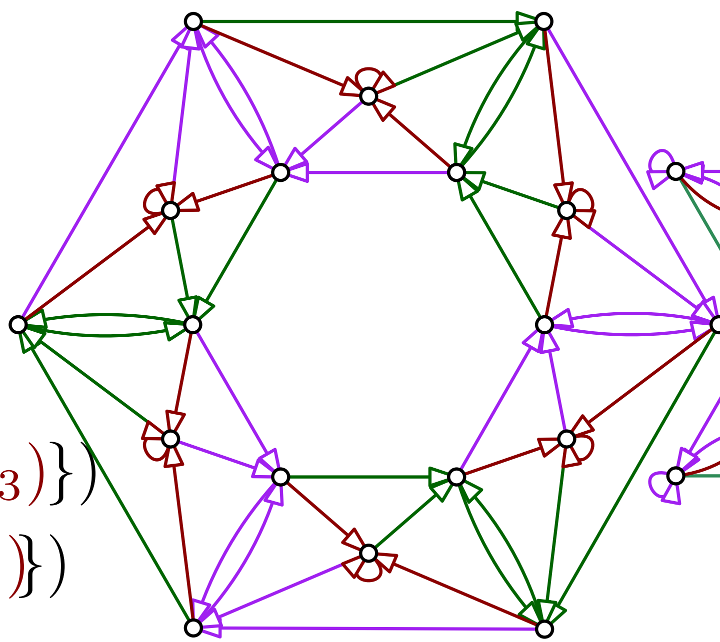
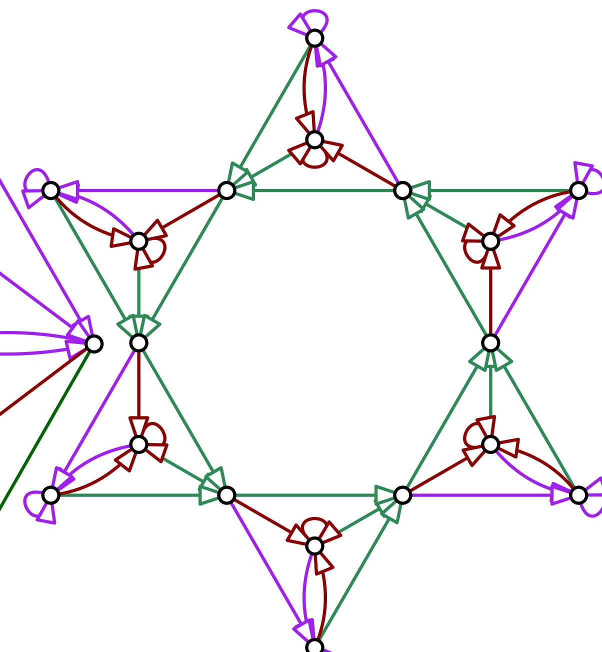
Planar semigroups?

Thm [K, Knauer '16]: the planar *right groups* are
$R_i \times \mathbb{Z}_n$,$R_i \times D_n$, $R_i \times A_4$,
$R_i \times S_4$, $R_i \times A_5$, R_{i+1}, $i \leq 3$





Cay$(D_3 \times R_3, \{(a, r_1),(b, r_2),(e, r_3)\})$

Cay$(\mathbb{Z}_6 \times R_3, \{(1, r_1),(0, r_2),(0, r_3)\})$

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Cor: the planar Cayley graphs of groups are 1-skeleta of:

Platonic solids
(n-gon), cube, tetrahedron, octahedron, icosahedron
Archimedean solids
n-prism, cuboctahedron truncated cube, truncated octahedron, rhombicuboctahedron, truncated dodecahedron, truncated icosahedron, rhombicosidodecahedron, truncated icosidodecahedron, n-antiprism, snub cube, snub dodecahedron

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Archimedean solids missing: dodecahedron

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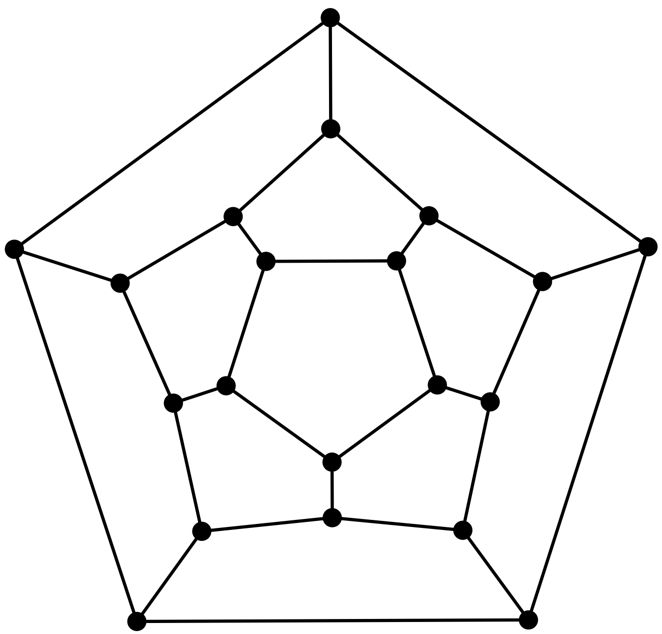
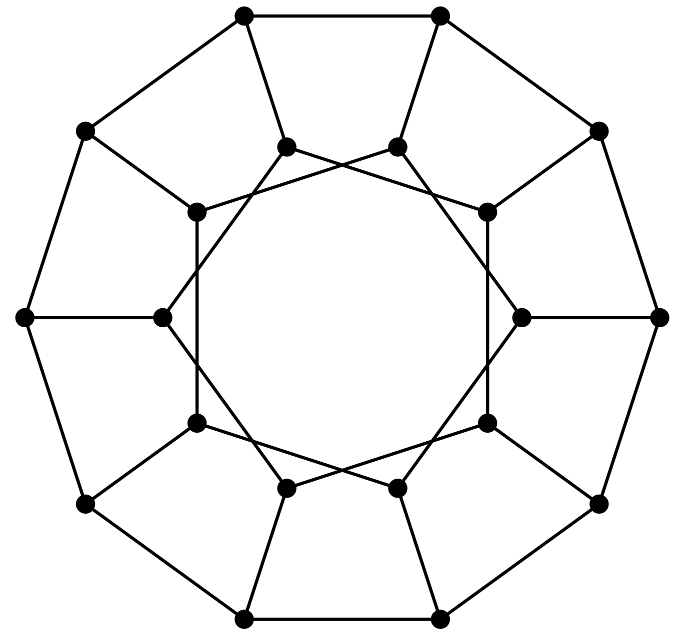
Quest[K,Knauer '16]:
are they underlying Cayley graphs of monoids?

The dodecahedron

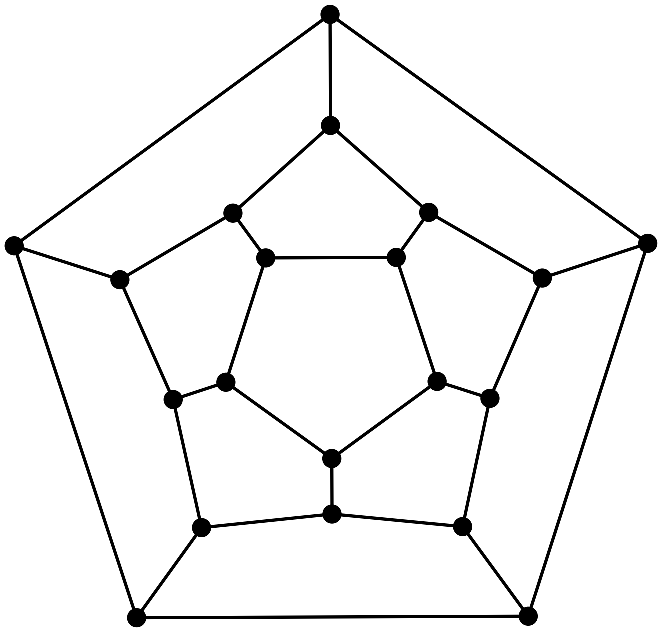


$$\cong$$



The dodecahedron

generalized Petersen graph $G(10, 2)$



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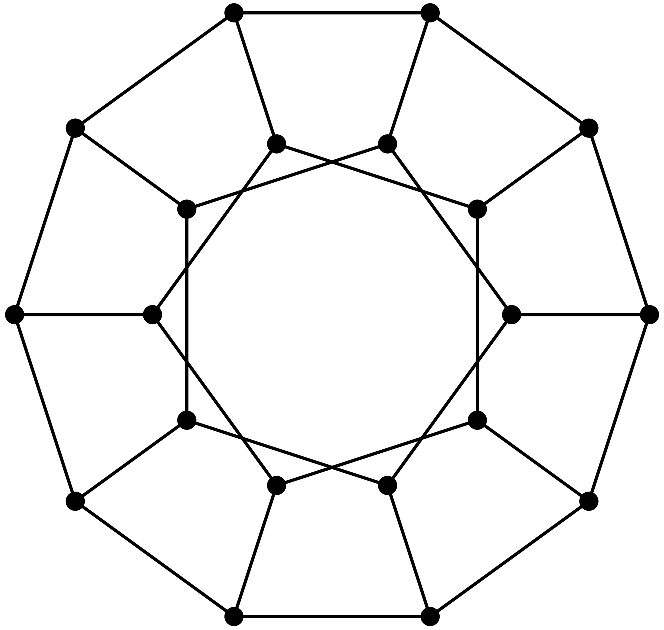

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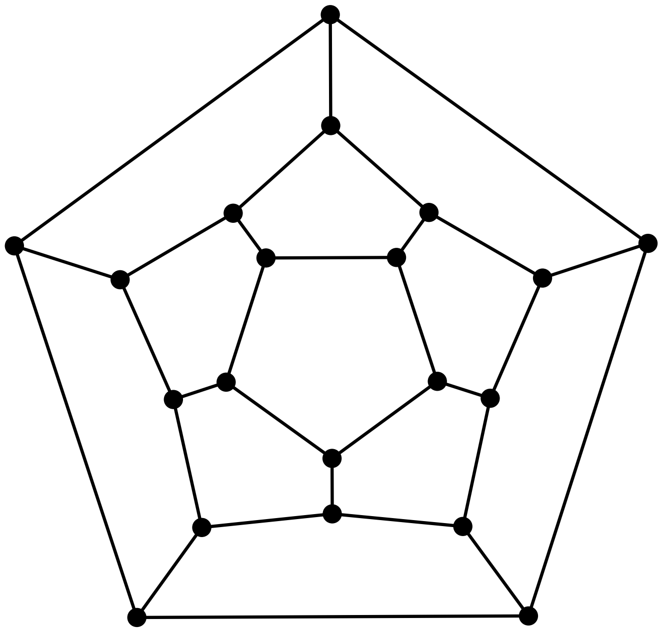

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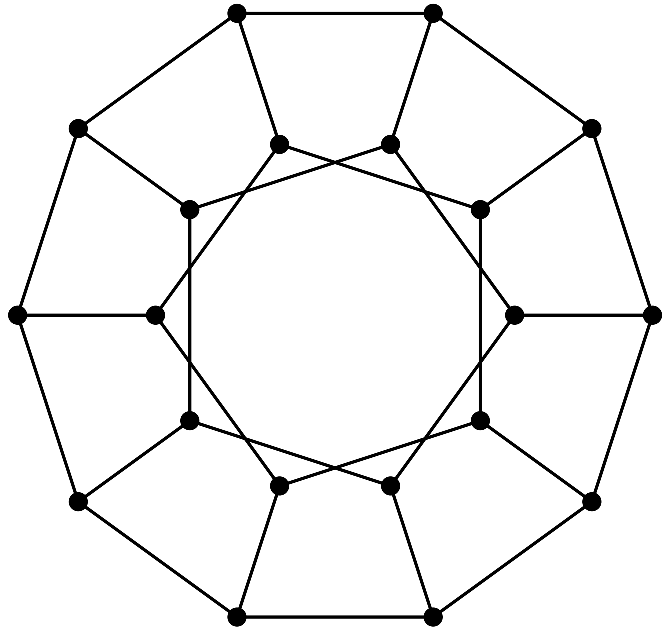
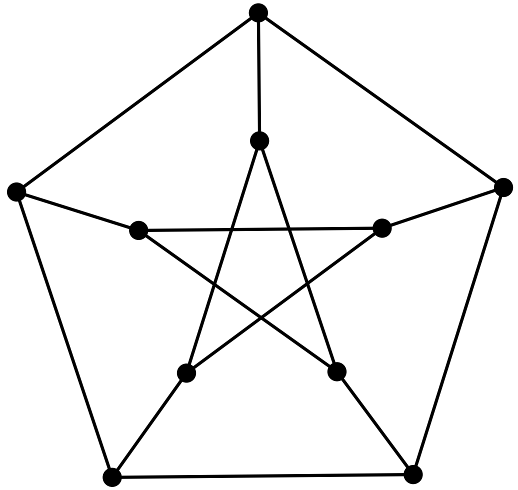


(generalized) Petersen graph $G(5, 2)$



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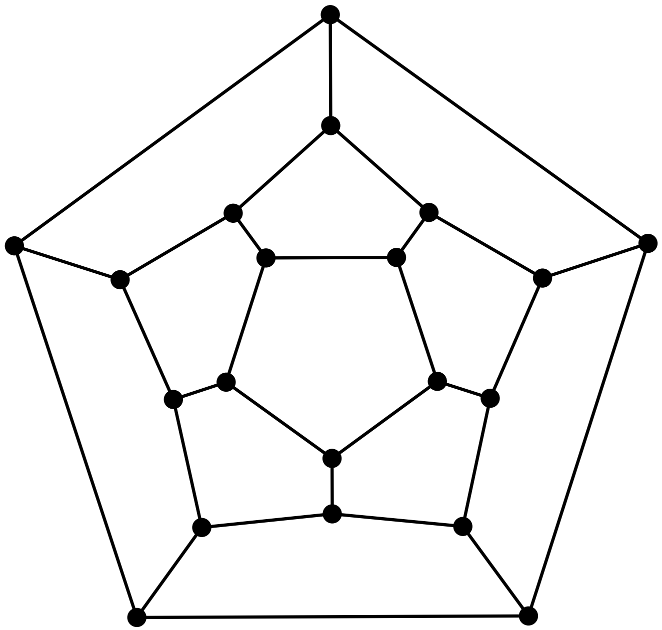

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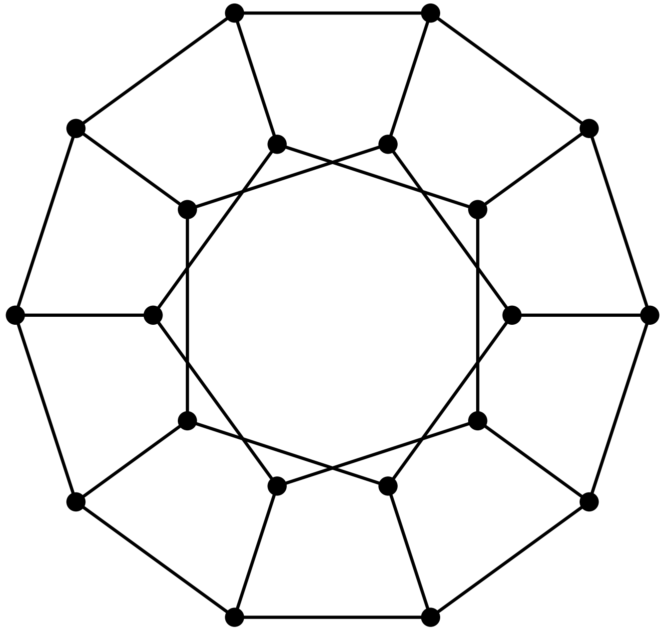

Quest[Mathoverflow '21]:
is the Petersen graph a "Cayley graph" of some more general group-like structure?

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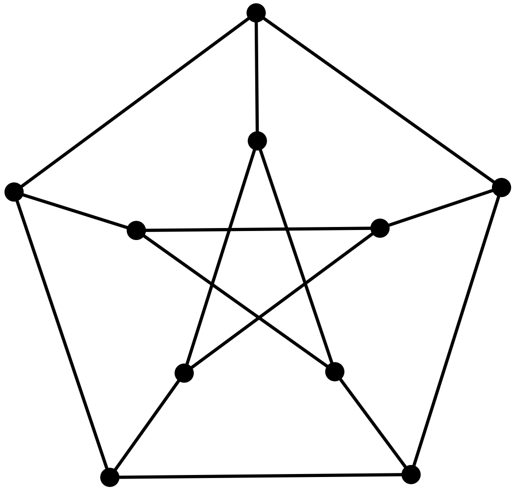

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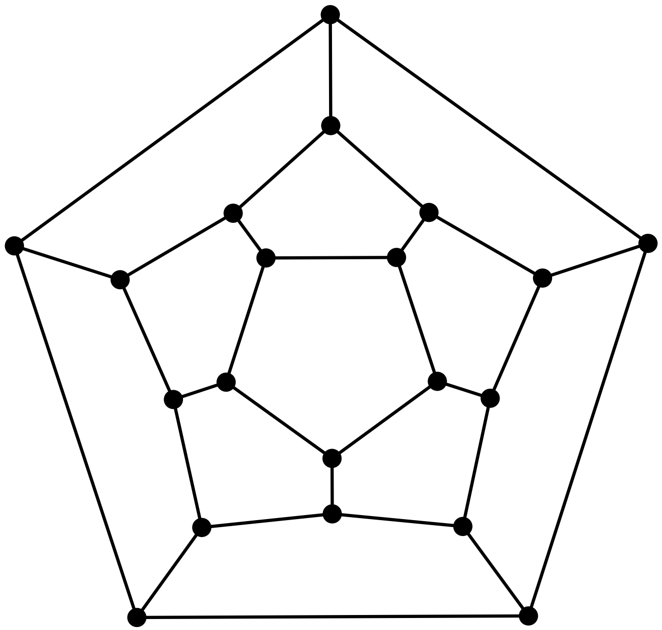

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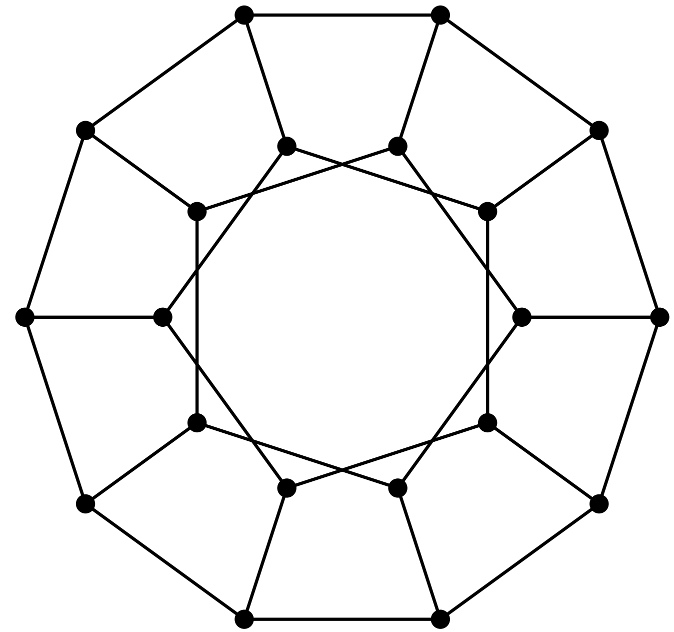

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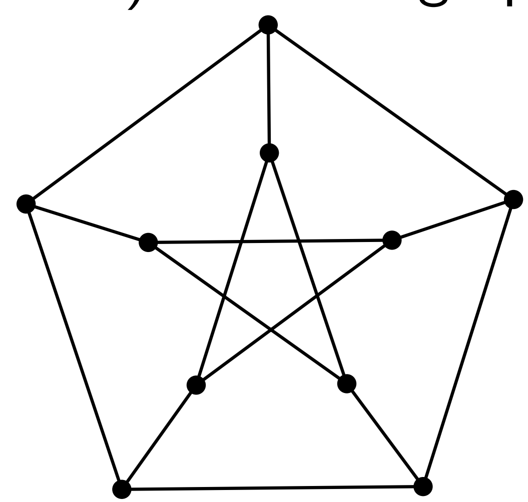

how about $G(n,k)$?

Generalized Petersen Graphs

Thm[Nedal, Škoviera '95, Lovrečič Saražin '97]:
$G(n,k)$ is Cayley of group iff $k^2 = 1 \mod n$.

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$$x(i,\ell_j) = (x+i \mod \tfrac{n}{\gcd(n,k)}, \ell_{x+j \mod \gcd(n,k)})$$
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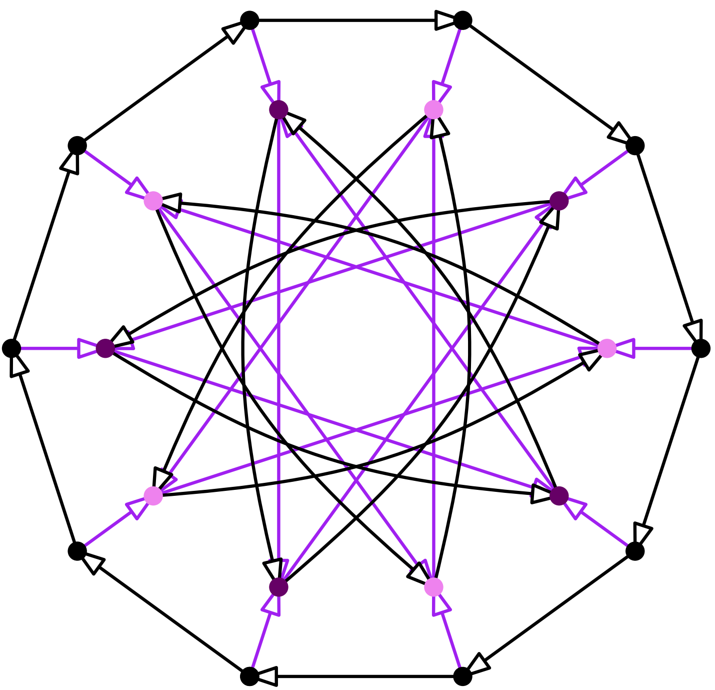

$\mathrm{Cay}(S, \{1, (1,\ell_0)\})$

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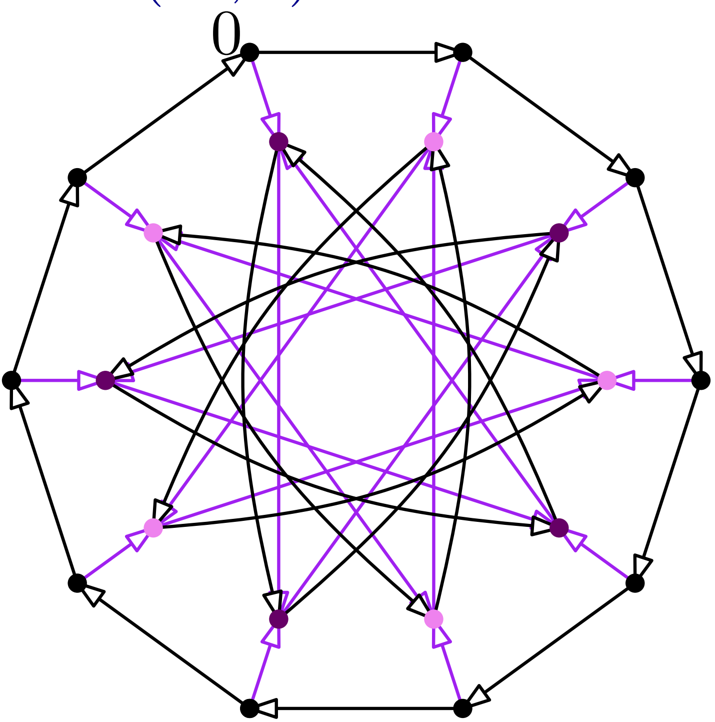

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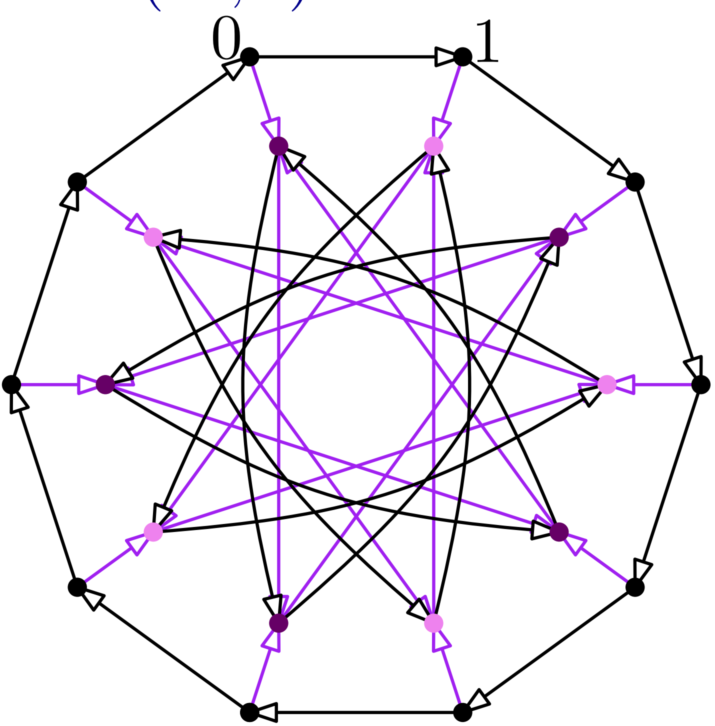

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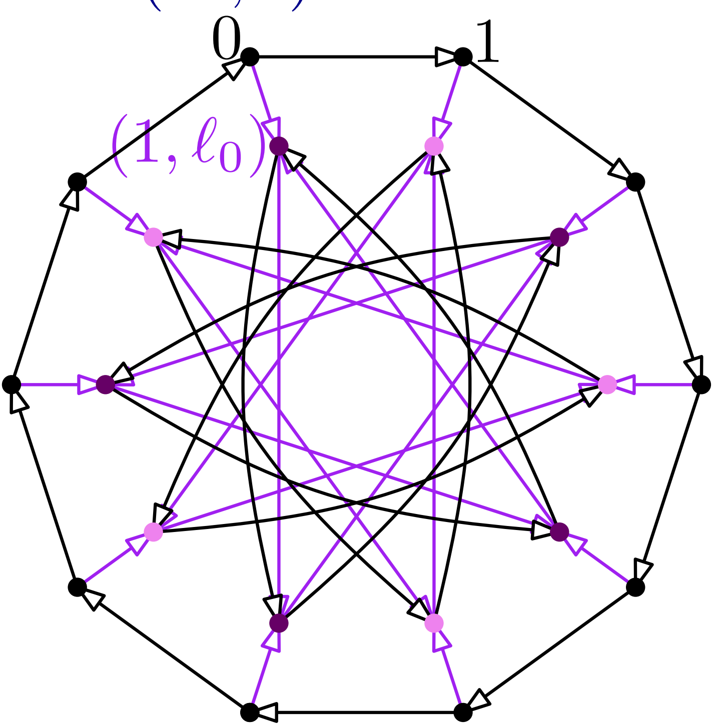

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$$(i,\ell_j)x = (x + i \mod \frac{n}{\gcd(n,k)}, \ell_j)$$

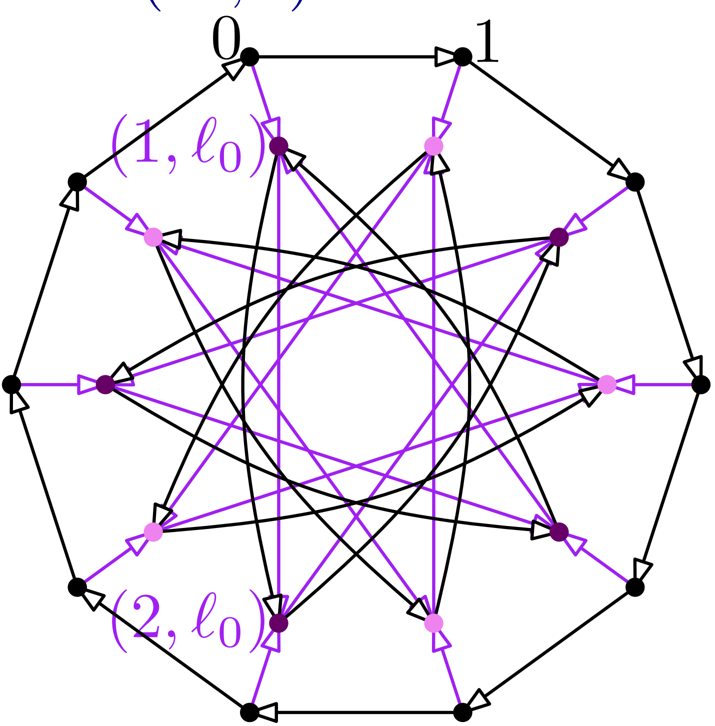
Generalized Petersen Graphs

Thm[Nedal, Škoviera '95, Lovrečič Saražin '97]:
$G(n,k)$ is Cayley of group iff $k^2 = 1 \mod n$.

Thm[García-Marco, K '24]:
$G(n,k)$ is Cayley of monoid if $k = \pm 1 \mod \frac{n}{\gcd(n,k)}$.

proof: $S = \mathbb{Z}_n \cup (\mathbb{Z}_{\frac{n}{\gcd(n,k)}} \times L_{\gcd(n,k)}) = \mathbb{Z}_{10} \cup (\mathbb{Z}_5 \times L_2)$

$G(10,4)$



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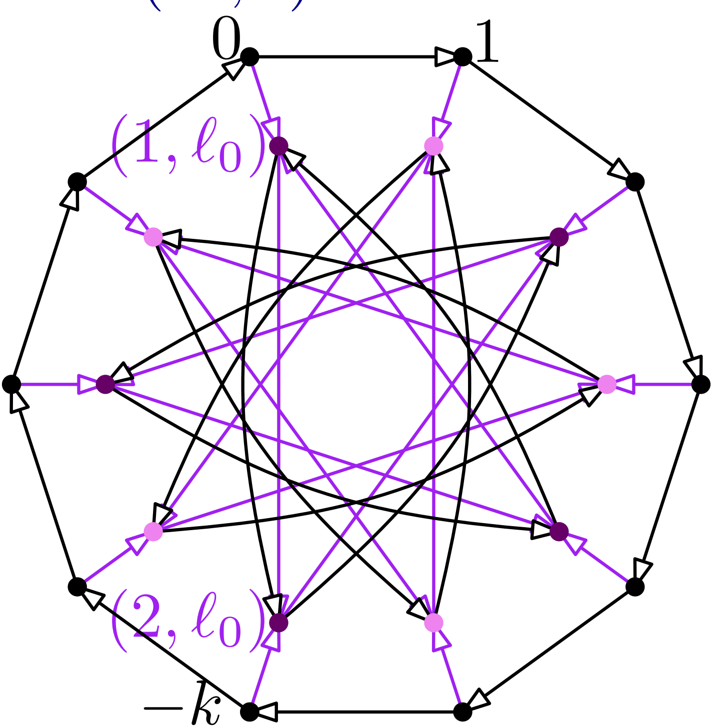

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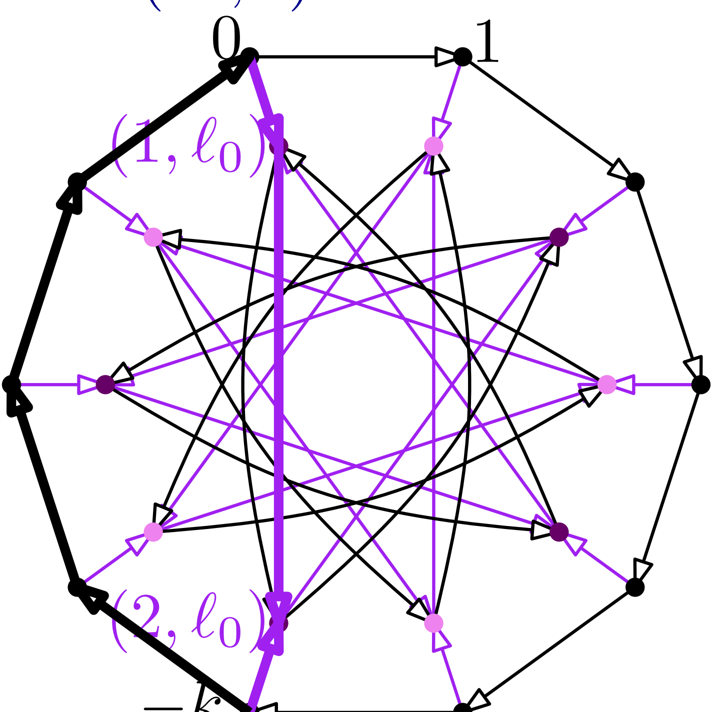

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$$-k + 1 = 2 \mod \tfrac{n}{\gcd(n,k)}$$
$$k = -1 \mod \tfrac{n}{\gcd(n,k)}$$

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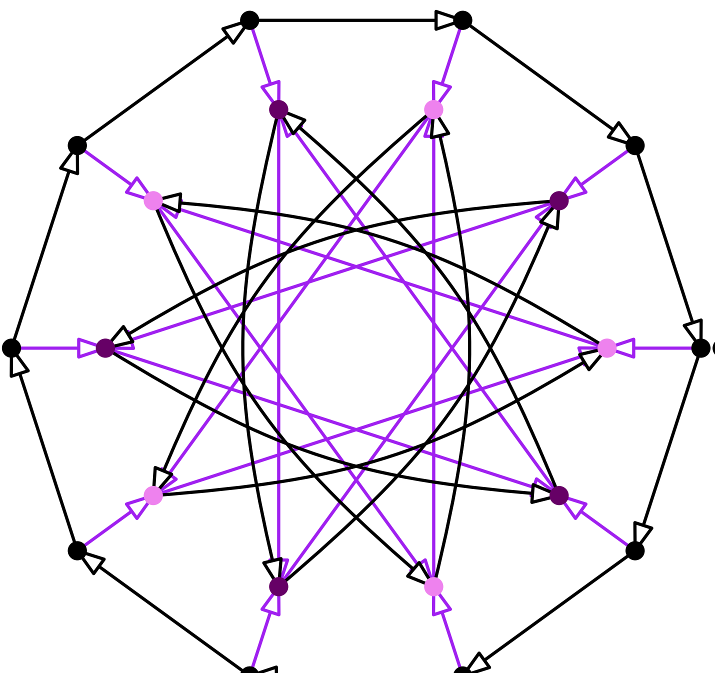
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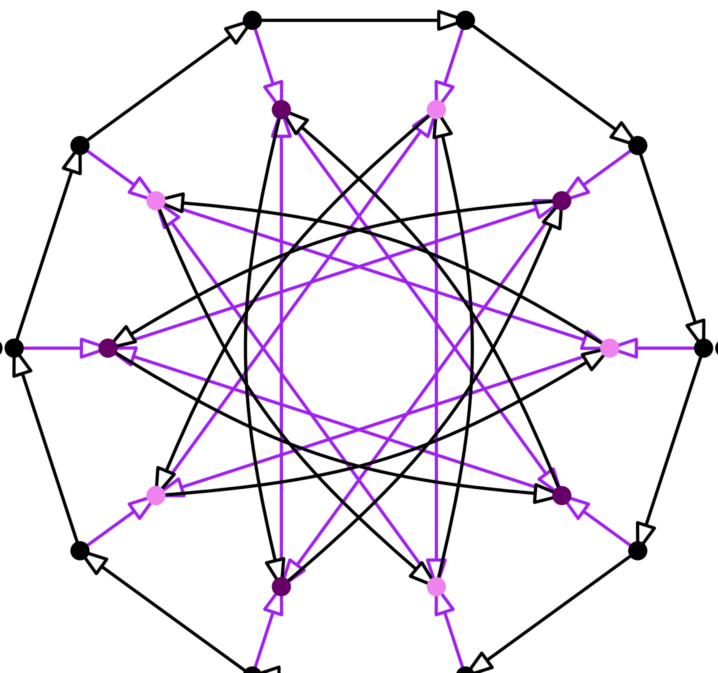
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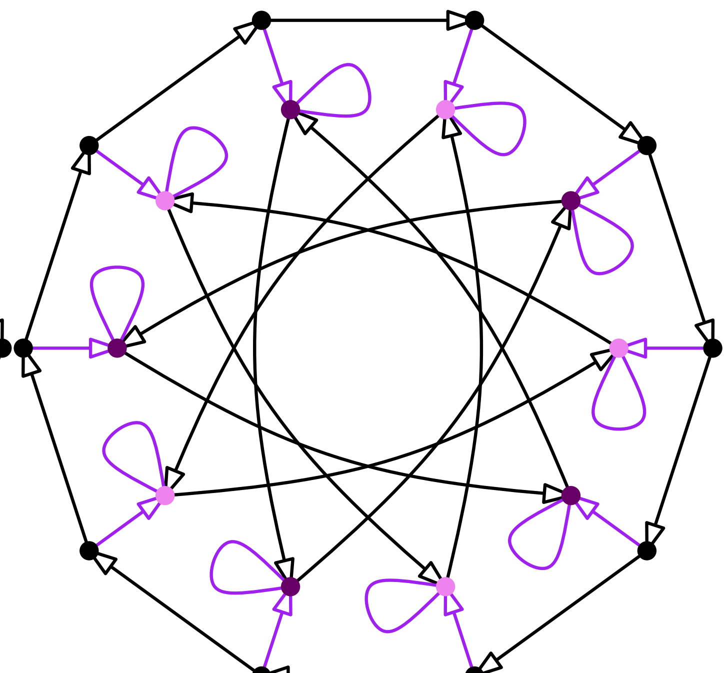
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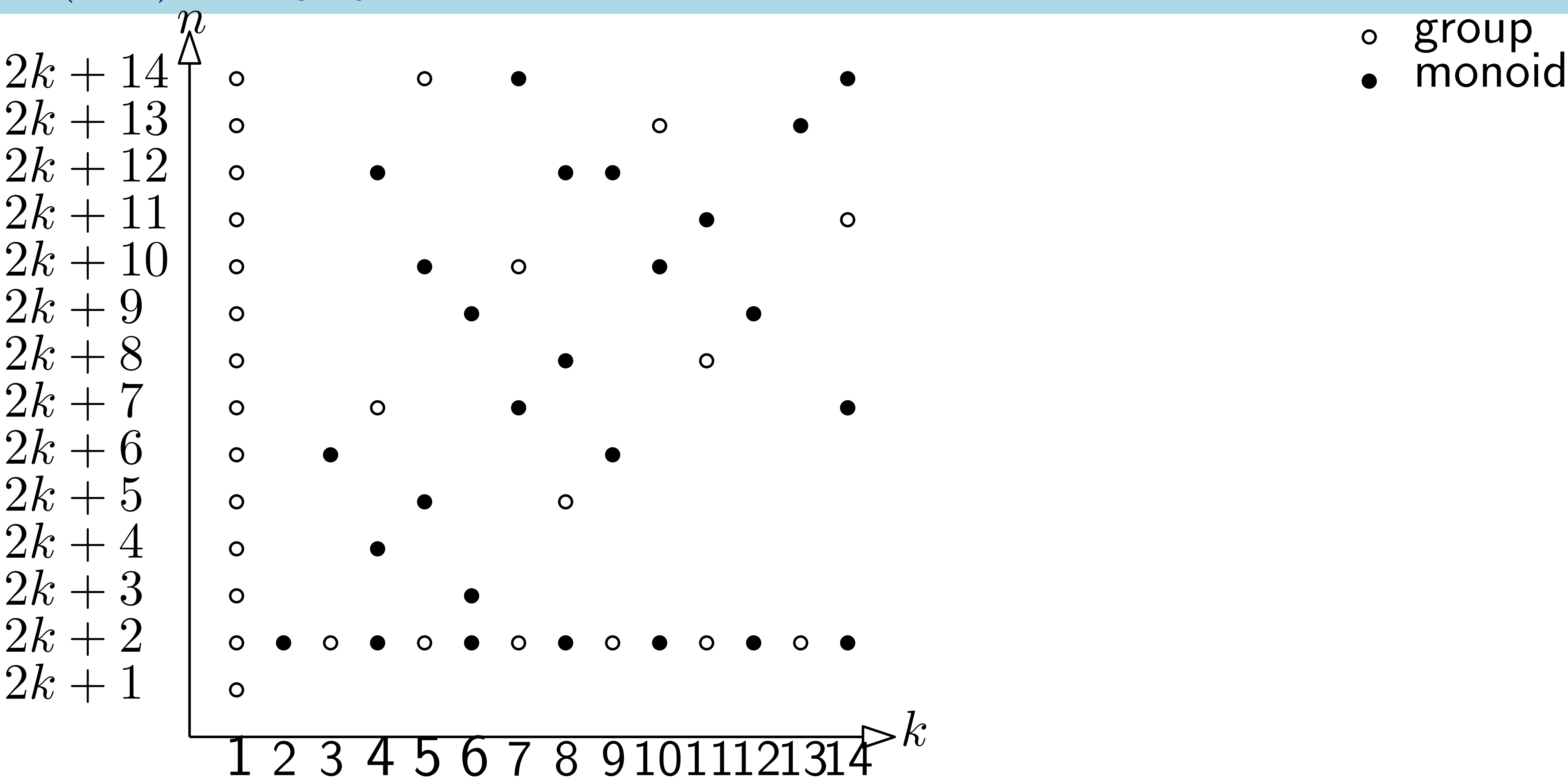

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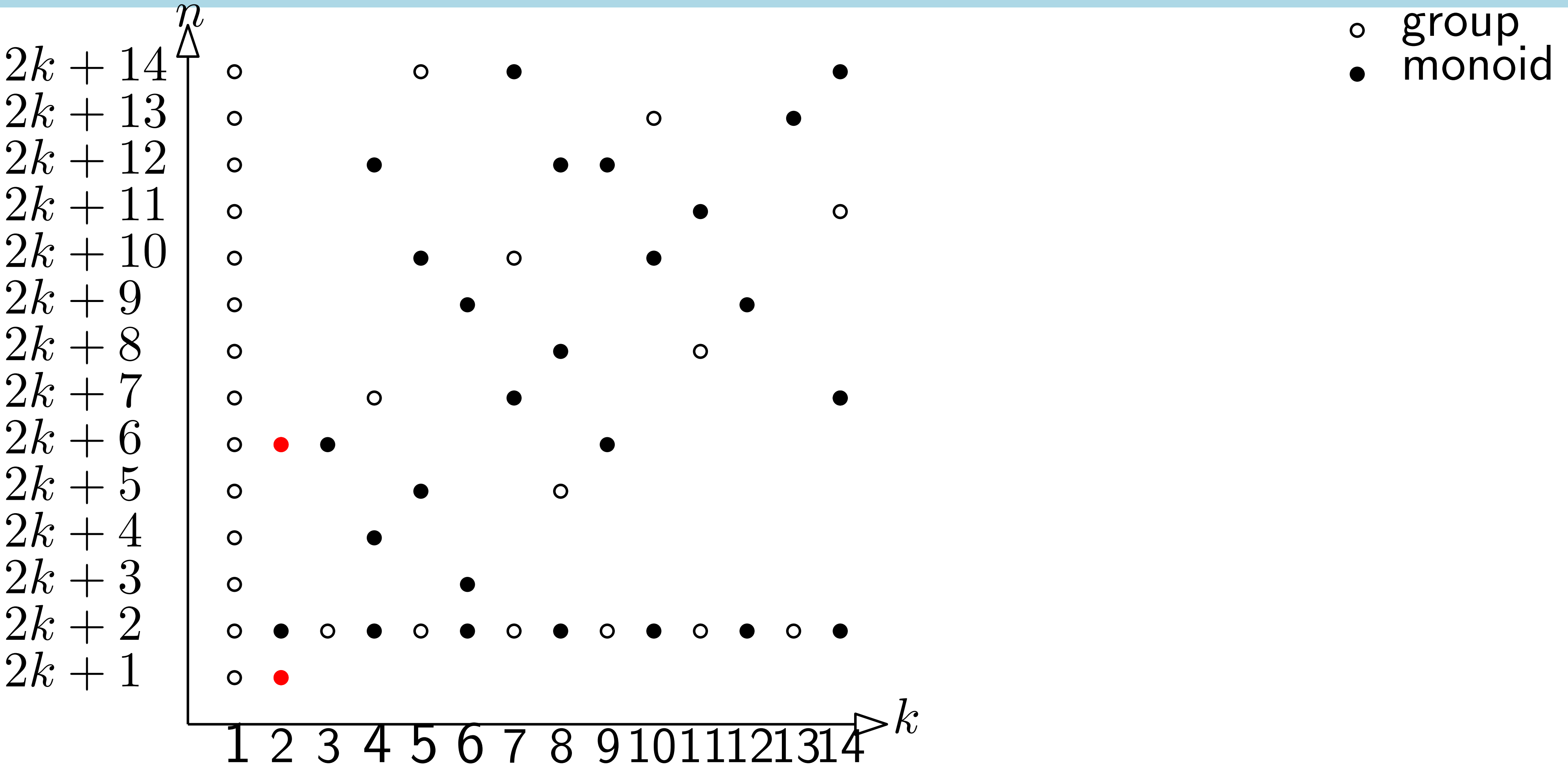
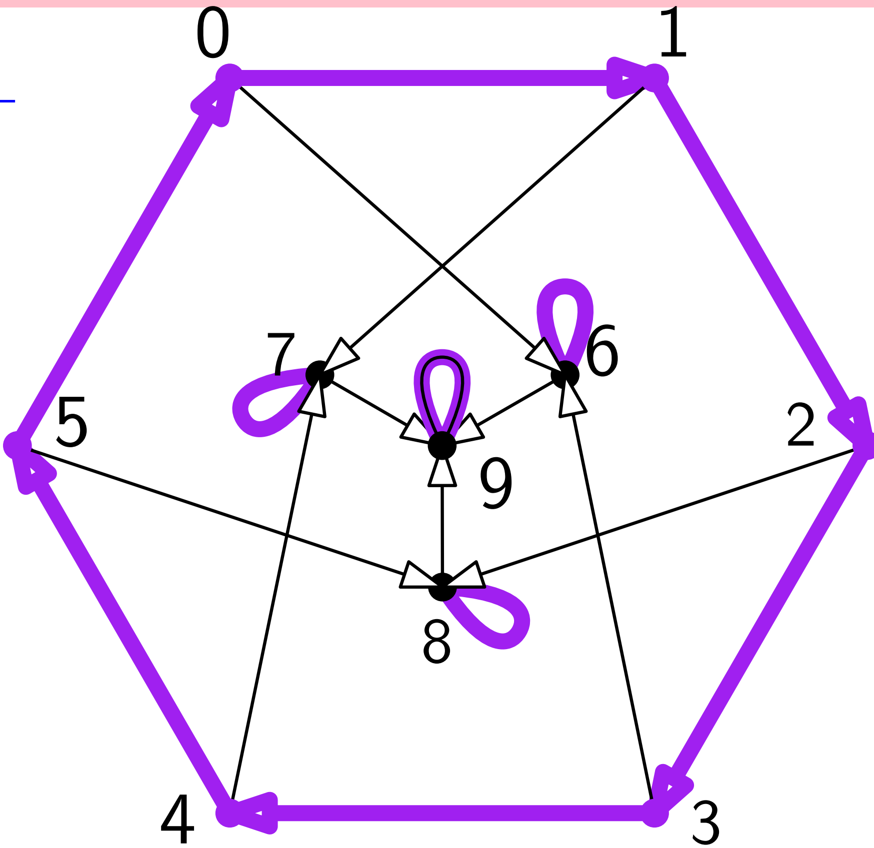


The Petersen graph

M	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	0	7	8	6	9
2	2	3	4	5	0	1	8	6	7	9
3	3	4	5	0	1	2	6	7	8	9
4	4	5	0	1	2	3	7	8	6	9
5	5	0	1	2	3	4	8	6	7	9
6	6	6	6	6	6	6	9	9	9	9
7	7	7	7	7	7	7	9	9	9	9
8	8	8	8	8	8	8	9	9	9	9
9	9	9	9	9	9	9	9	9	9	9



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0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	0	7	8	6	9
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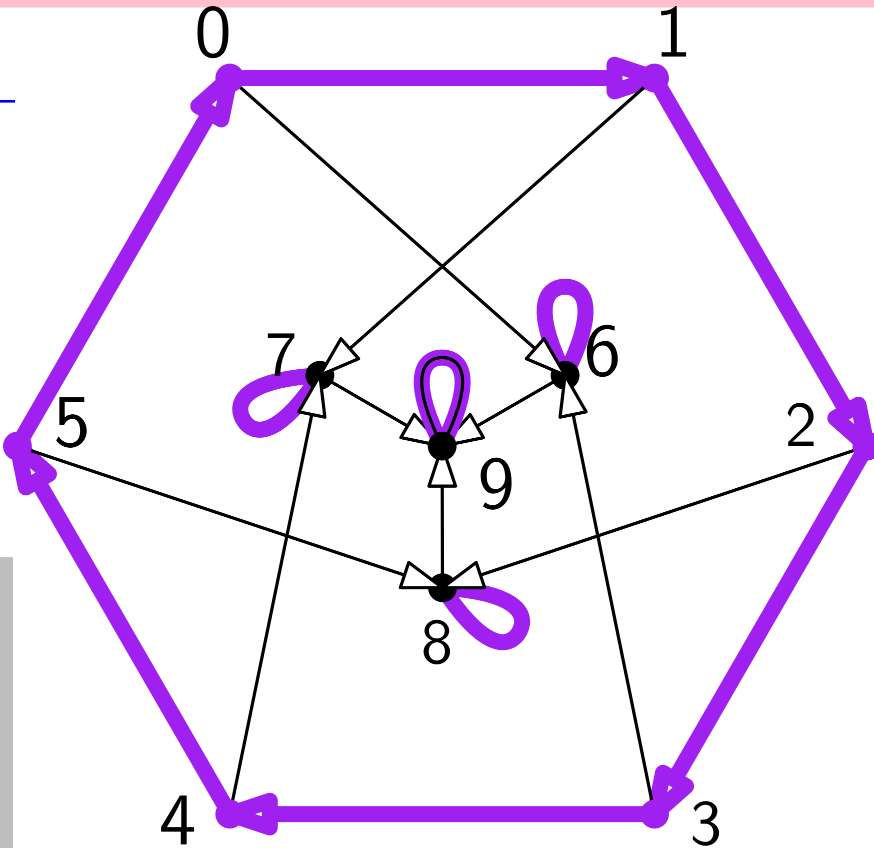
3	3	4	5	0	1	2	6	7	8	9
4	4	5	0	1	2	3	7	8	6	9
5	5	0	1	2	3	4	8	6	7	9
6	6	6	6	6	6	6	9	9	9	9
7	7	7	7	7	7	7	9	9	9	9
8	8	8	8	8	8	8	9	9	9	9
9	9	9	9	9	9	9	9	9	9	9



exploiting the (less obvious) symmetry $\mathbb{Z}_6 < \mathrm{Aut}(G(5,2))$

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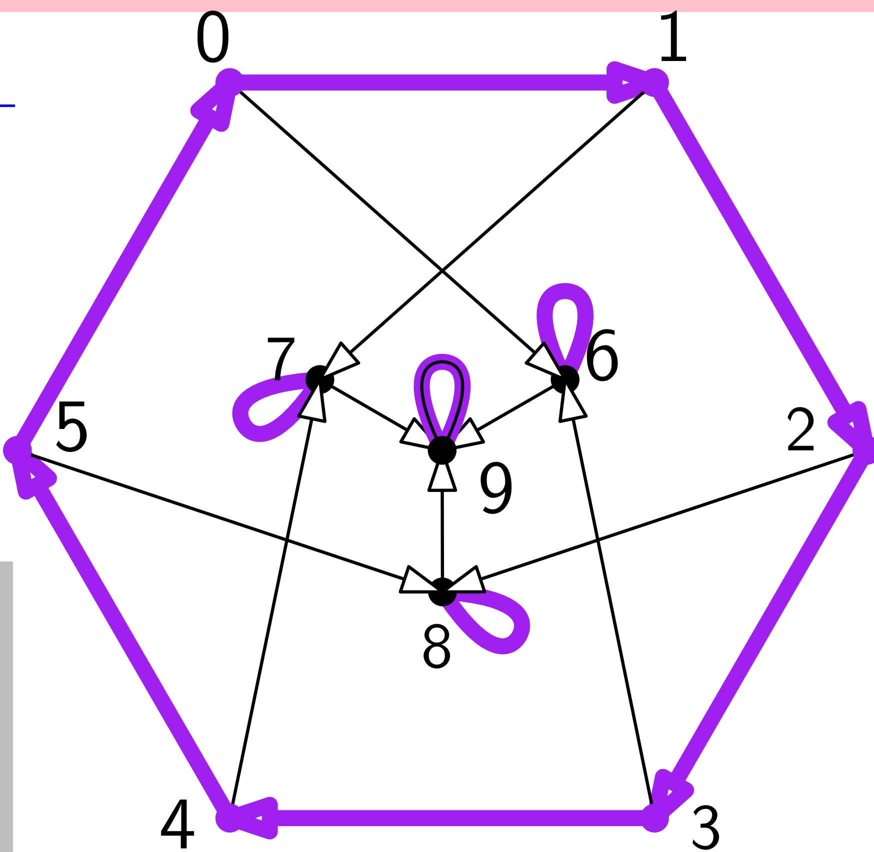
M	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	0	7	8	6	9
2	2	3	4	5	0	1	8	6	7	9
3	3	4	5	0	1	2	6	7	8	9
4	4	5	0	1	2	3	7	8	6	9
5	5	0	1	2	3	4	8	6	7	9
6	6	6	6	6	6	6	9	9	9	9
7	7	7	7	7	7	7	9	9	9	9
8	8	8	8	8	8	8	9	9	9	9
9	9	9	9	9	9	9	9	9	9	9



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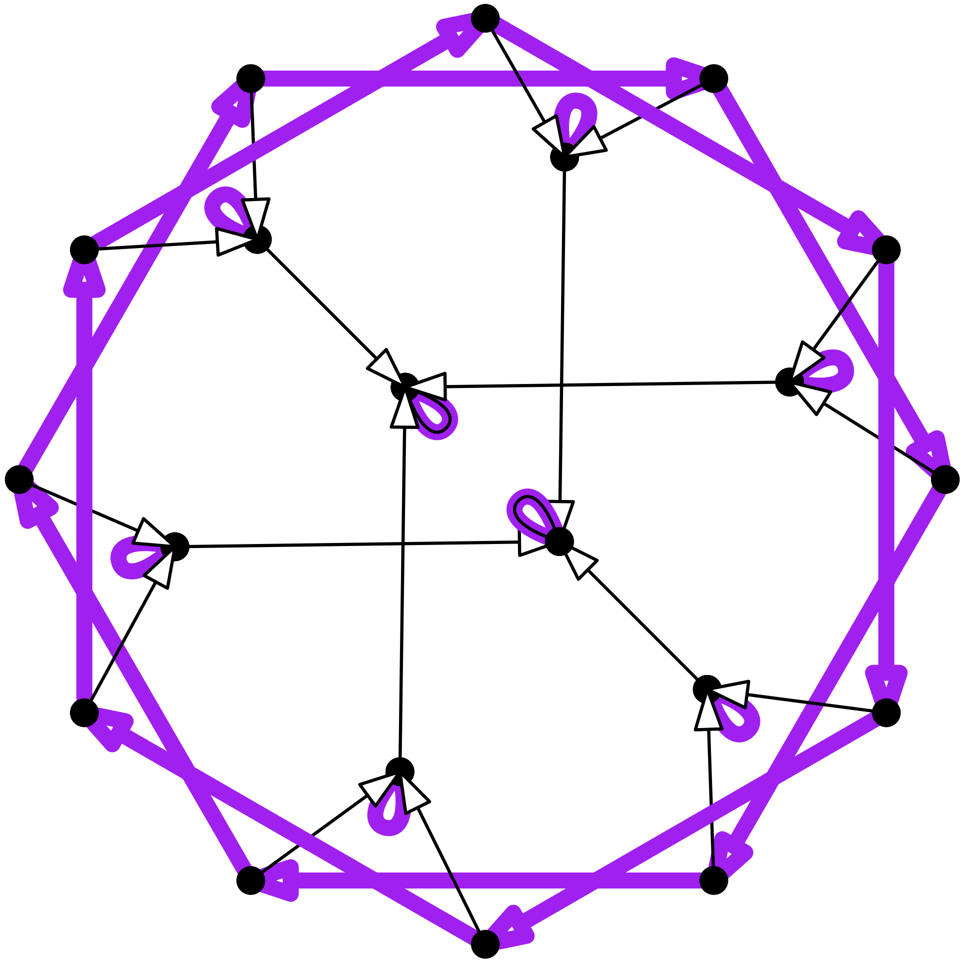
two more graphs

Desargues graph and Dodecahedron

$G(10, 3)$



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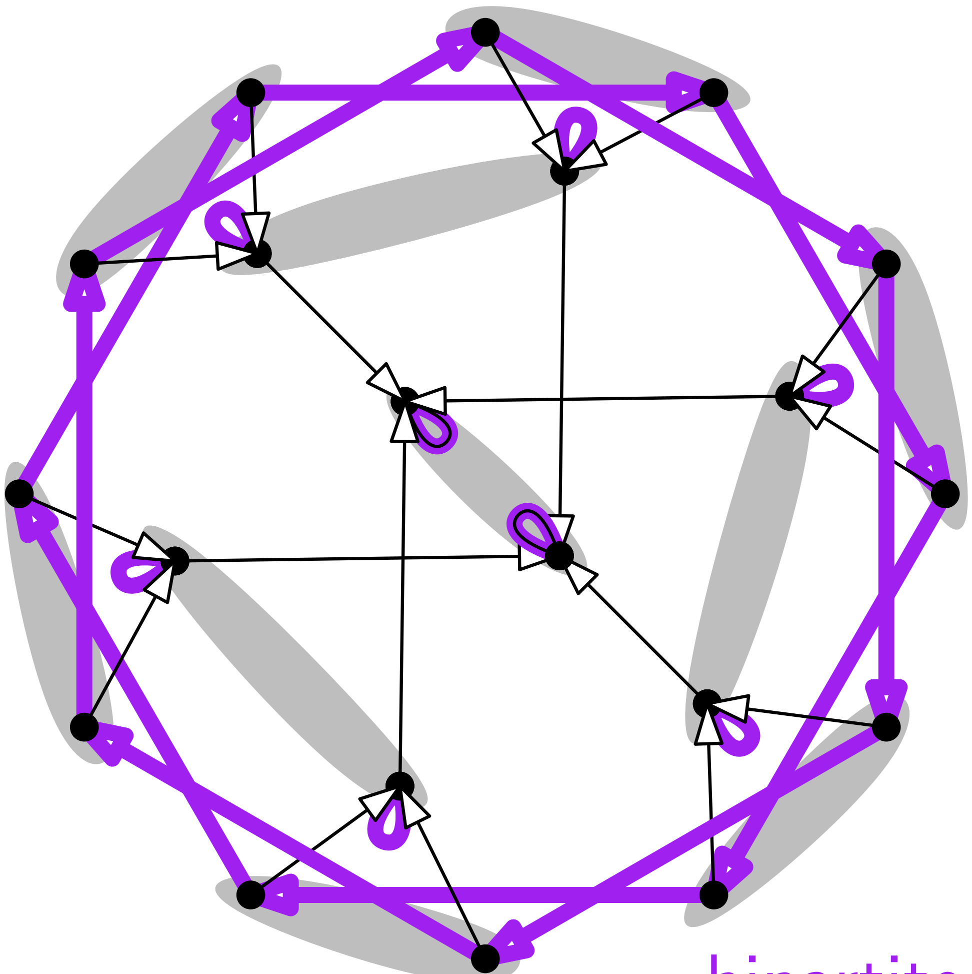
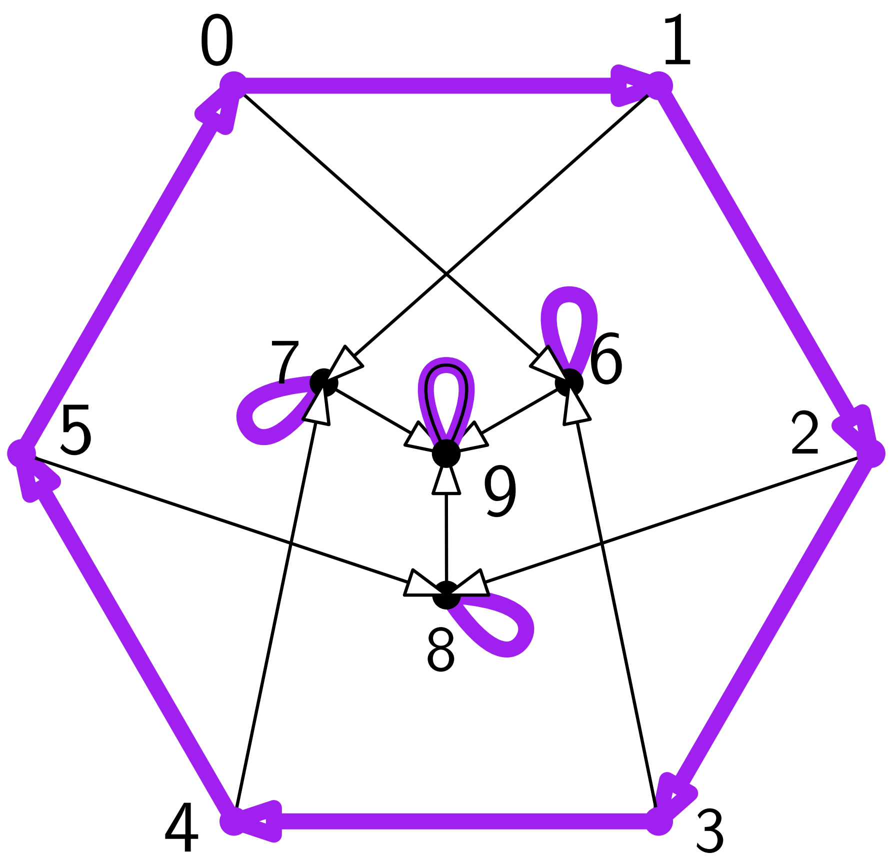


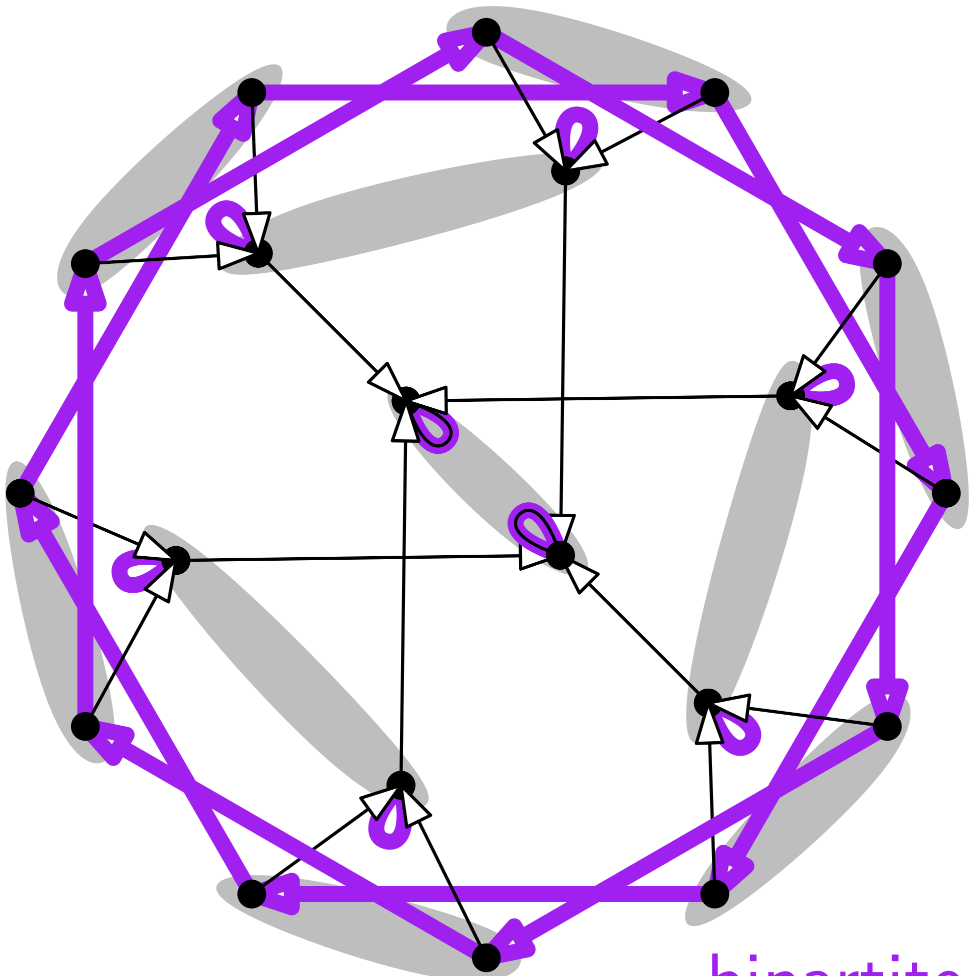
$G(5, 2)$



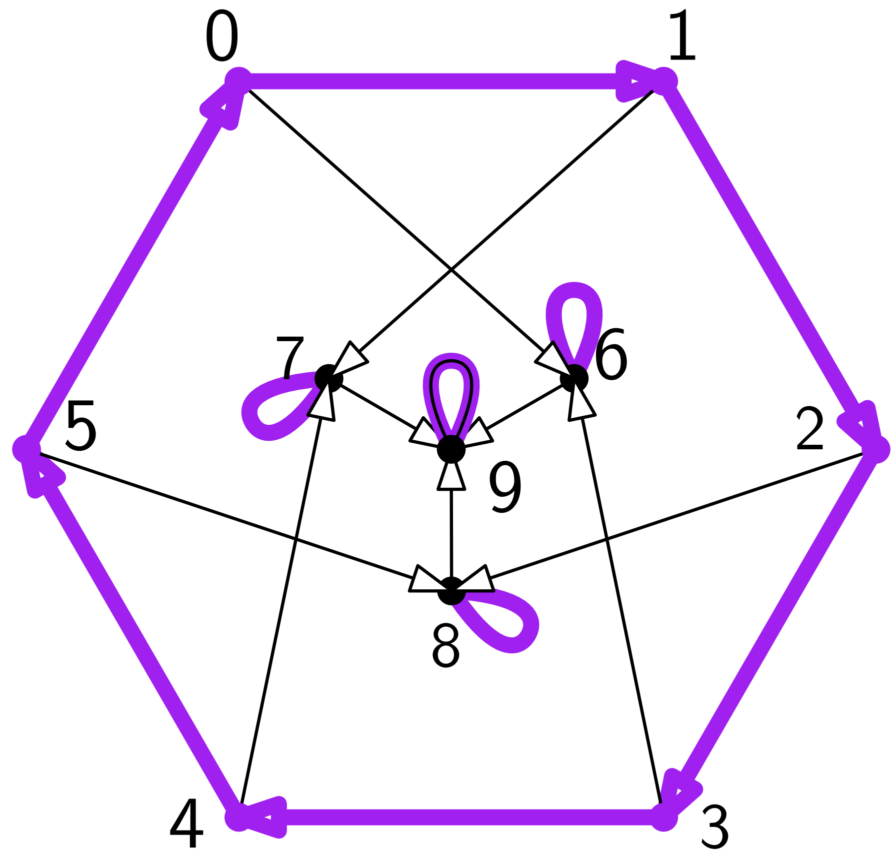
bipartite double cover

Desargues graph and Dodecahedron

$G(10,3)$



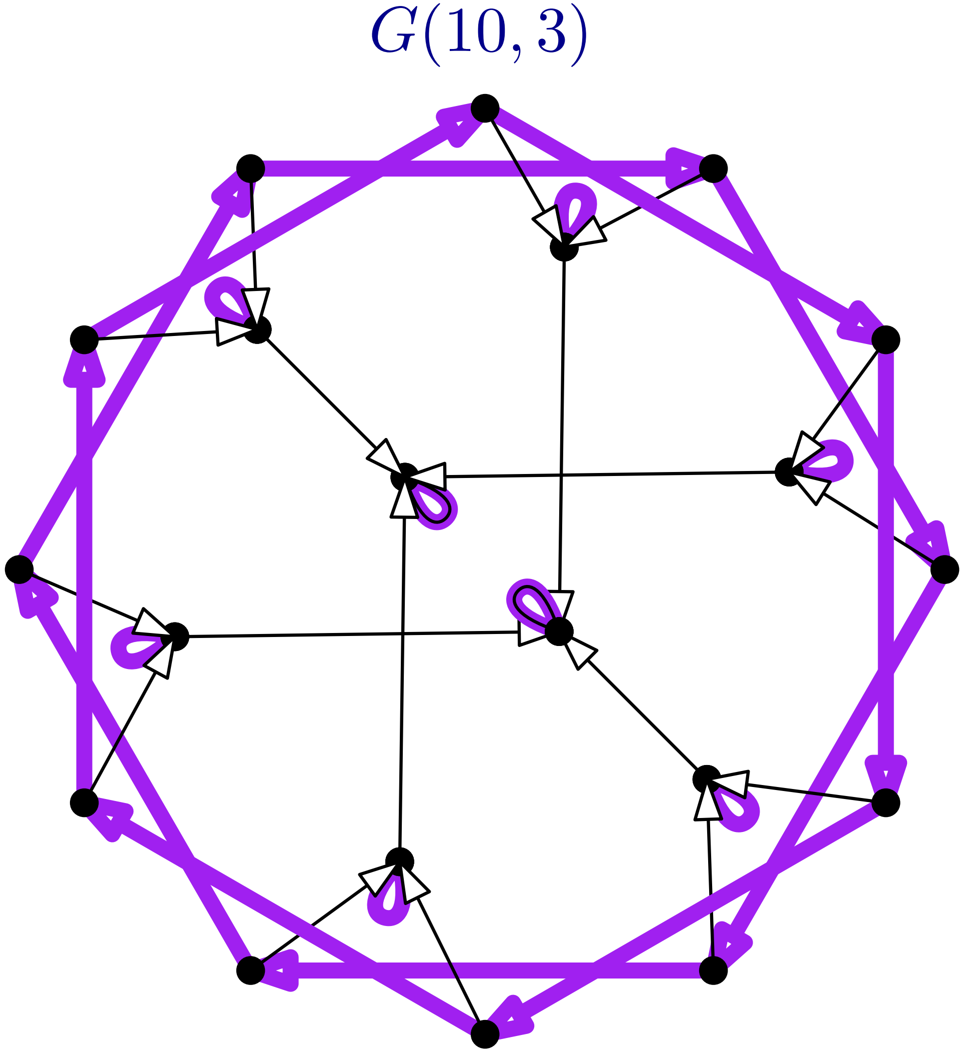
$G(5,2)$

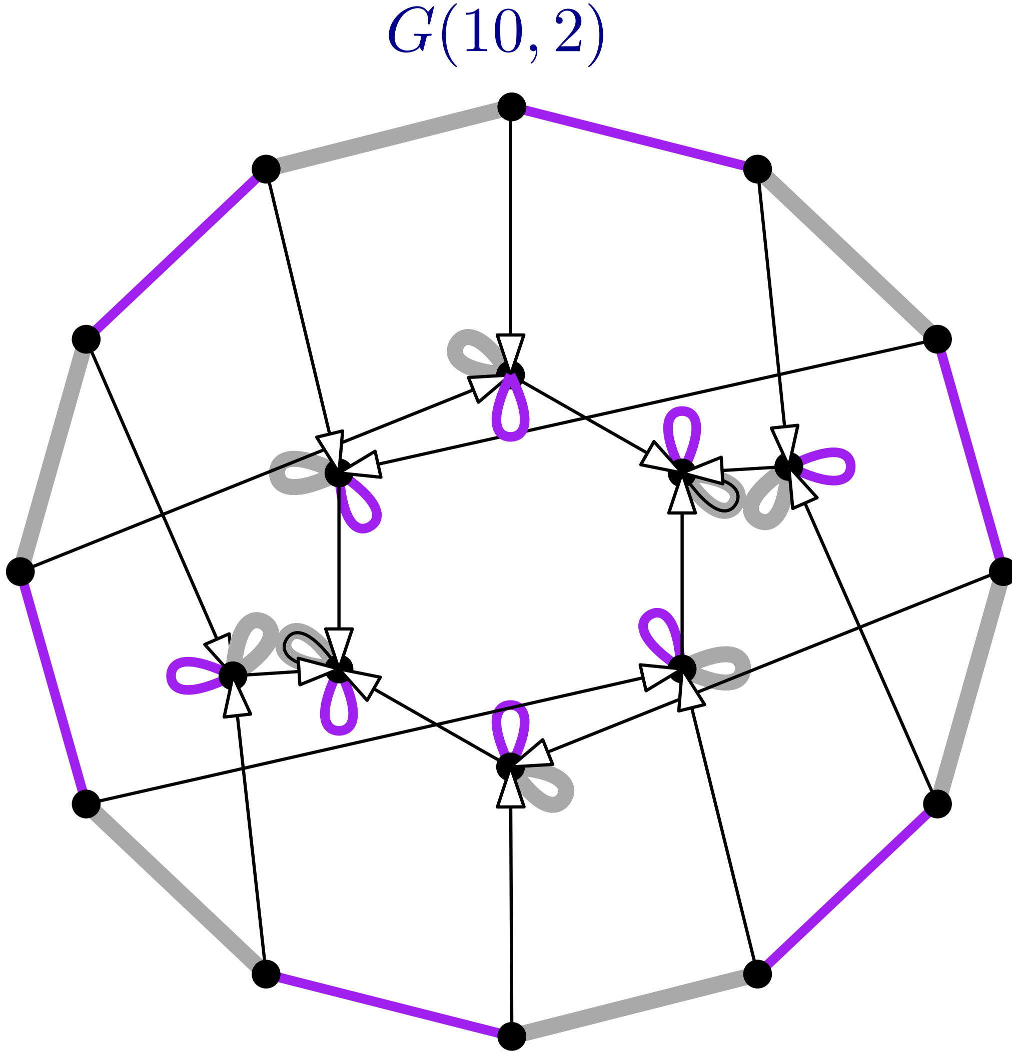


bipartite double cover

Cor[Krnc, Pisanski, '19]:
apart from this, the trick won't leave the family $k^2 = \pm k \mod n$

Desargues graph and Dodecahedron





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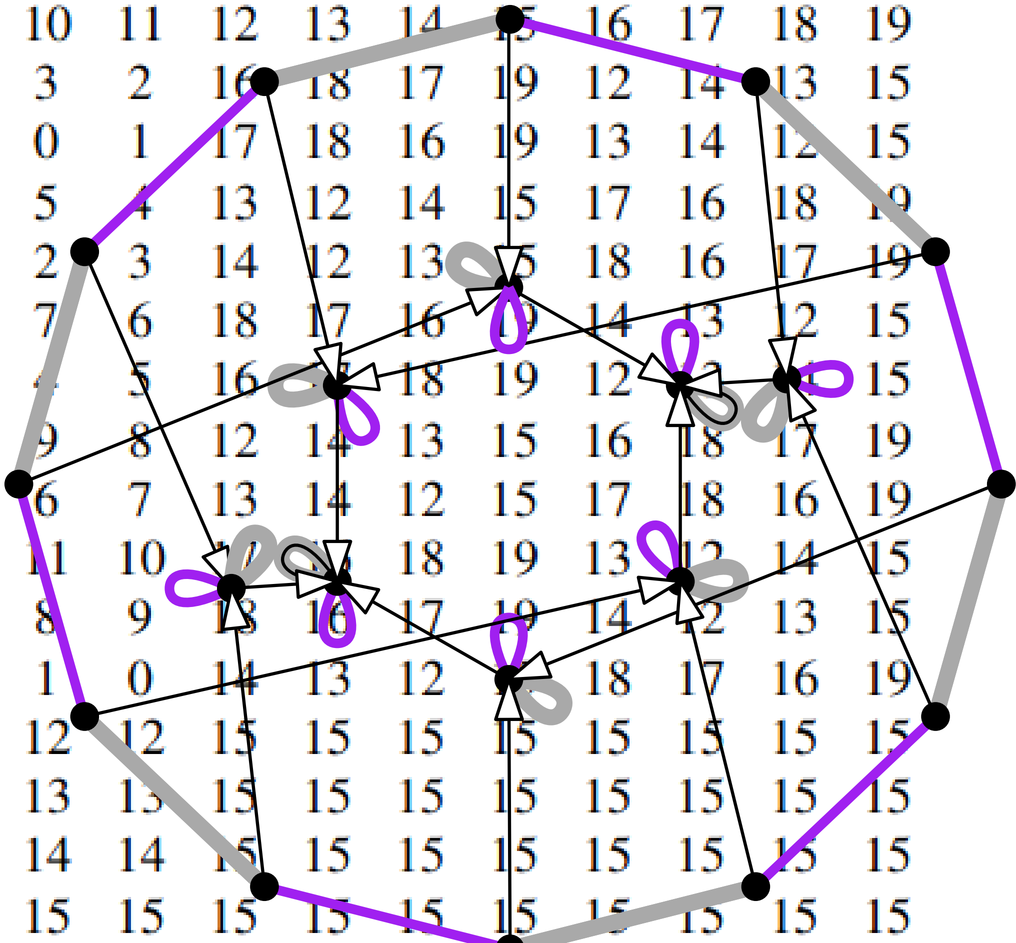

M	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1	0	11	10	9	8	7	6	5	4	3	2	16	18	17	19	12	14	13	15
2	2	3	4	5	6	7	8	9	10	11	0	1	17	18	16	19	13	14	12	15
3	3	2	1	0	11	10	9	8	7	6	5	4	13	12	14	15	17	16	18	19
4	4	5	6	7	8	9	10	11	0	1	2	3	14	12	13	15	18	16	17	19
5	5	4	3	2	1	0	11	10	9	8	7	6	18	17	16	19	14	13	12	15
6	6	7	8	9	10	11	0	1	2	3	4	5	16	[illegible]	18	19	12	[illegible]	[illegible]	15
7	7	6	5	4	3	2	1	0	11	10	9	8	12	14	13	15	16	18	17	19
8	8	9	10	11	0	1	2	3	4	5	6	7	13	14	12	15	17	18	16	19
9	9	8	7	6	5	4	3	2	1	0	11	10	[illegible]	[illegible]	18	19	13	12	14	15
10	10	11	0	1	2	3	4	5	6	7	8	9	18	16	17	19	14	12	13	15
11	11	10	9	8	7	6	5	4	3	2	1	0	14	13	12	[illegible]	18	17	16	19
12	12	12	12	12	12	12	12	12	12	12	12	12	15	15	15	15	15	15	15	15
13	13	13	13	13	13	13	13	13	13	13	13	13	15	15	15	15	15	15	15	15
14	14	14	14	14	14	14	14	14	14	14	14	14	15	15	15	15	15	15	15	15
15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15
16	16	16	16	16	16	16	16	16	16	16	16	16	19	19	19	19	19	19	19	19
17	17	17	17	17	17	17	17	17	17	17	17	17	19	19	19	19	19	19	19	19
18	18	18	18	18	18	18	18	18	18	18	18	18	19	19	19	19	19	19	19	19
19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19

exploiting the (less obvious) symmetry $D_6 < \mathrm{Aut}(G(10,2))$

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M	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1	0	11	10	9	8	7	6	5	4	3	2	16	18	17	19	12	14	13	15
2	2	3	4	5	6	7	8	9	10	11	0	1	17	18	16	19	13	14	12	15
3	3	2	1	0	11	10	9	8	7	6	5	4	13	12	14	15	17	16	18	19
4	4	5	6	7	8	9	10	11	0	1	2	3	14	12	13	15	18	16	17	19
5	5	4	3	2	1	0	11	10	9	8	7	6	18	17	16	19	14	13	12	15
6	6	7	8	9	10	11	0	1	2	3	4	5	16	17	18	19	12	13	14	15
7	7	6	5	4	3	2	1	0	11	10	9	8	12	14	13	15	16	18	17	19
8	8	9	10	11	0	1	2	3	4	5	6	7	13	14	12	15	17	18	16	19
9	9	8	7	6	5	4	3	2	1	0	11	10	17	16	18	19	13	12	14	15
10	10	11	0	1	2	3	4	5	6	7	8	9	18	16	17	19	14	12	13	15
11	11	10	9	8	7	6	5	4	3	2	1	0	14	13	12	15	18	17	16	19
12	12	12	12	12	12	12	12	12	12	12	12	12	15	15	15	15	15	15	15	15
13	13	13	13	13	13	13	13	13	13	13	13	13	15	15	15	15	15	15	15	15
14	14	14	14	14	14	14	14	14	14	14	14	14	15	15	15	15	15	15	15	15
15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15
16	16	16	16	16	16	16	16	16	16	16	16	16	19	19	19	19	19	19	19	19
17	17	17	17	17	17	17	17	17	17	17	17	17	19	19	19	19	19	19	19	19
18	18	18	18	18	18	18	18	18	18	18	18	18	19	19	19	19	19	19	19	19
19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19

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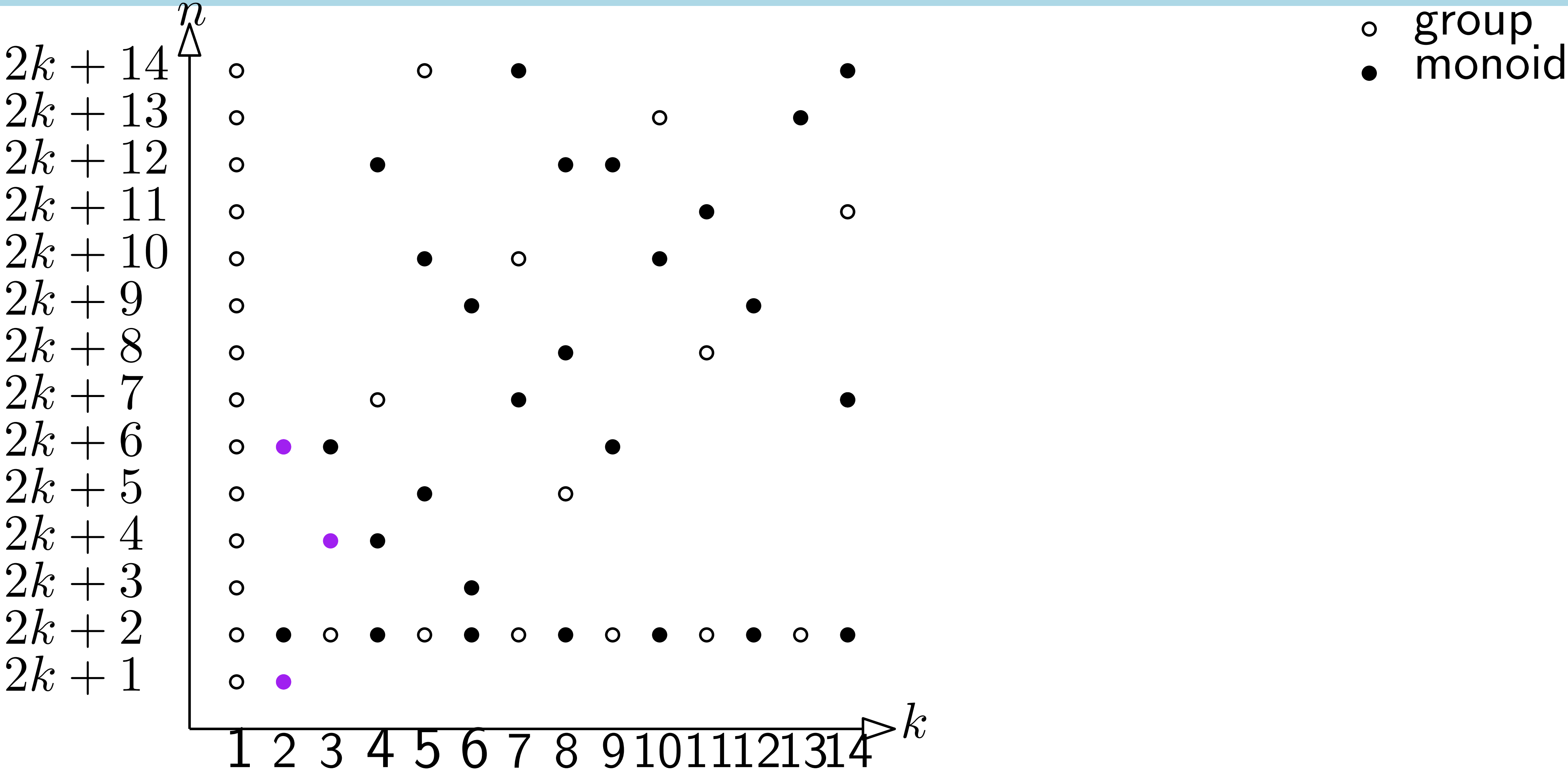

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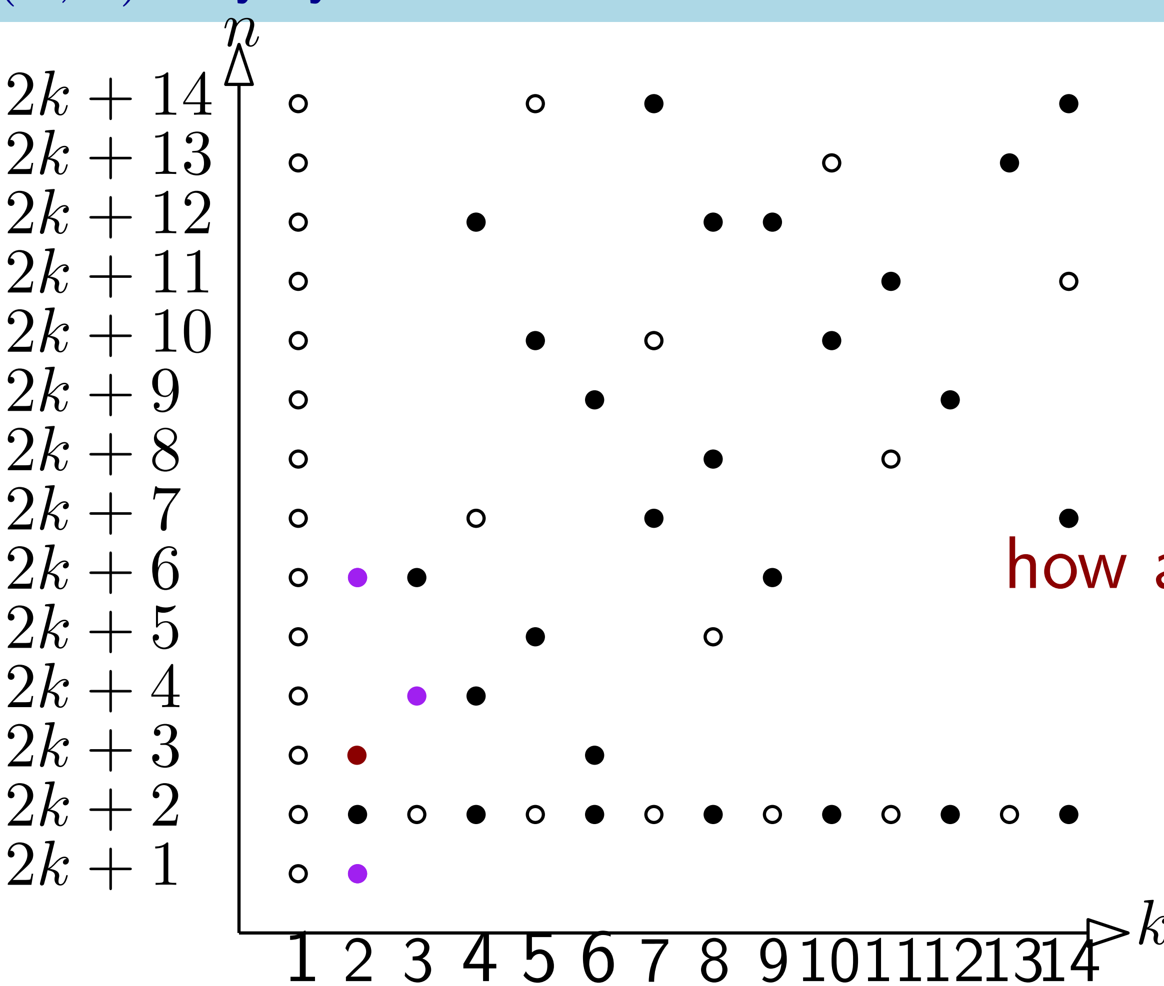

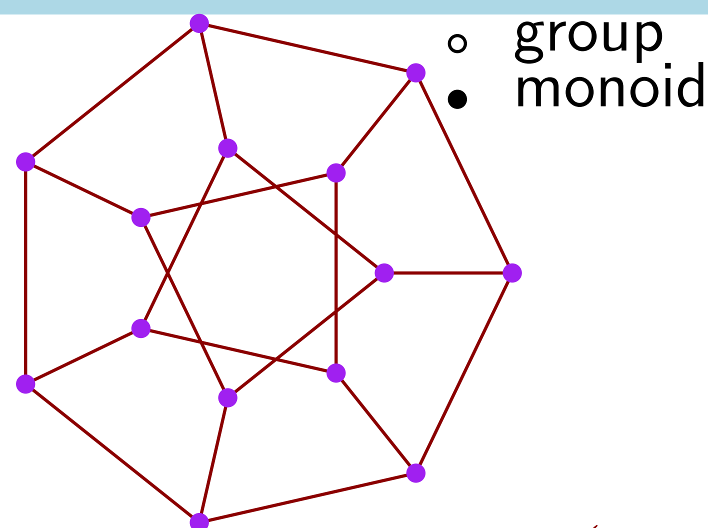

how about the rest, e.g. $G(7,2)$?

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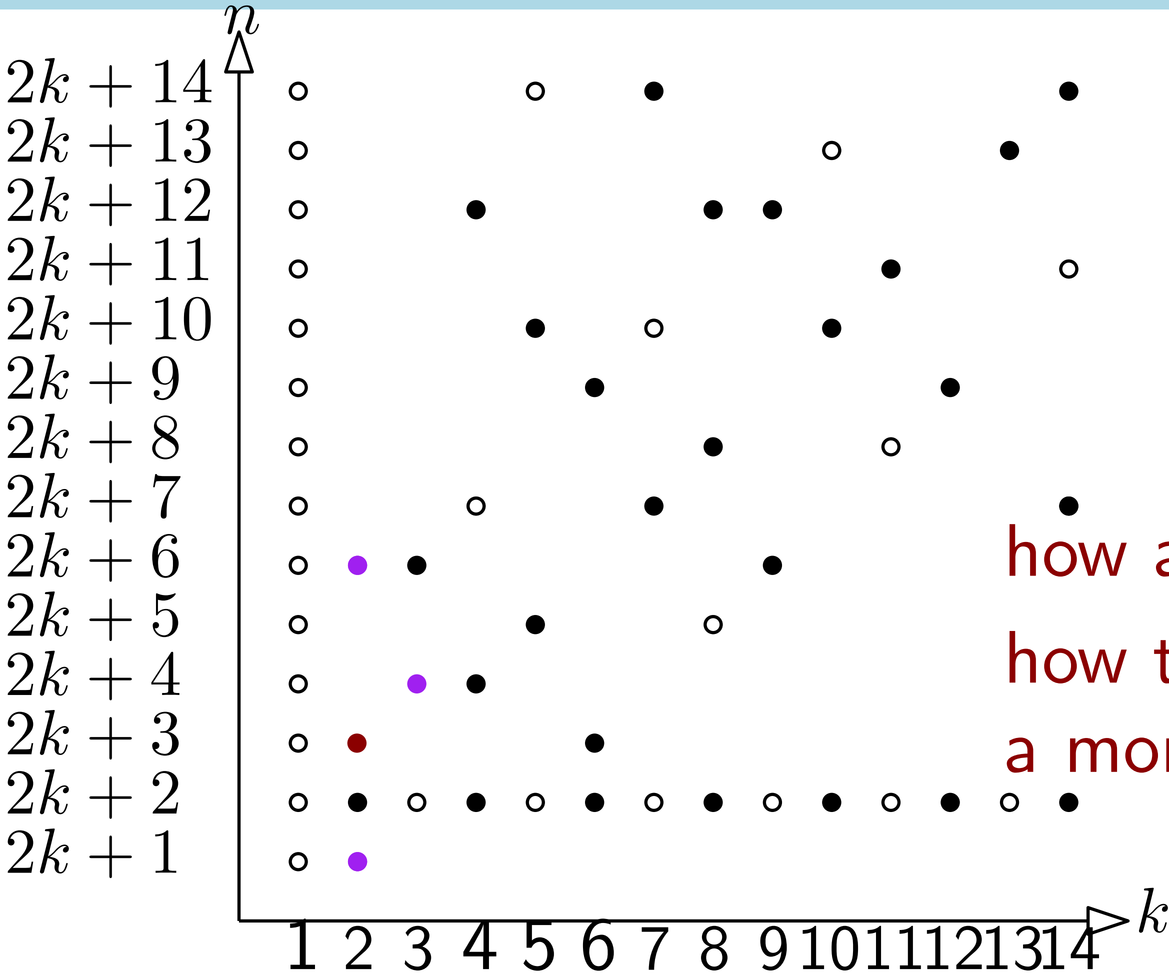

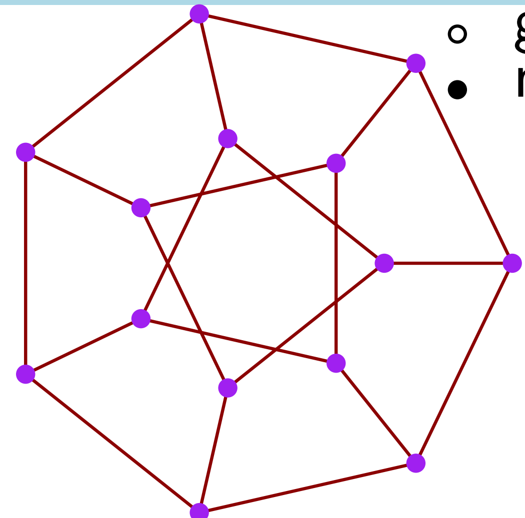

∘ group
• monoid

how about the rest, e.g. $G(7,2)$?

how to prove/check if a graph is a monoid or semigroup graph?

Cayley graphs

(finite) set S with binary operation $\cdot : S \times S \to S$

associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in S$ $\rightsquigarrow$ semigroup

neutral element: $\exists_{e \in S} : a \cdot e = e \cdot a = a$ for all $a \in S$ $\rightsquigarrow$ monoid

inverse element: $\forall_{a \in S} \exists_{a^{-1} \in S} : a \cdot a^{-1} = e$ $\rightsquigarrow$ group

order	#groups	# monoids	# semigroups
1	1	1	1
2	1	2	5
3	1	7	24
4	2	35	188
5	1	228	1915
6	2	2237	28634
7	1	31559	1627672
8	5	1668997	3684030417
9	2		105978177936292
10	2		
11	1		
12	5		
13	1		

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8	5	1668997	3684030417
9	2		105978177936292
10	2		
11	1		
12	5		
13	1		

trying all pairs $C \subseteq S$ is unfeasable

Restricting M

endomorphism:
$\varphi : V \to V$ such that:
$\{u,v\} \in E \Rightarrow \{\varphi(u), \varphi(v)\} \in E$
automorphism if bijective

$$
\begin{tikzcd}[column sep=small]
 & & \mathrm{End}(\underline{\mathrm{Cay}}^o(S,C)) \arrow[dl, dash] \arrow[dr, dash] & & \mathrm{Aut}(\underline{\mathrm{Cay}}(S,C)) \arrow[dl, dash] \\
 & \mathrm{End}(\mathrm{Cay}(S,C)) \arrow[dl, dash] \arrow[dr, dash] & & \mathrm{Aut}(\underline{\mathrm{Cay}}^o(S,C)) \arrow[dl, dash] & \\
\mathrm{End}(\mathrm{Cay}_{\mathrm{col}}(S,C)) \arrow[dr, dash] & & \mathrm{Aut}(\mathrm{Cay}(S,C)) \arrow[dl, dash] & & \\
 & \mathrm{Aut}(\mathrm{Cay}_{\mathrm{col}}(S,C)) & & &
\end{tikzcd}
$$

Restricting M

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left-multiplication with M is in $\mathrm{End}(\mathrm{Cay}_{\mathrm{col}}(M,C))$

$$\left(s \xrightarrow{c} sc = t\right) \xrightarrow{\lambda_m} \left(ms \xrightarrow{c} msc = mt\right)$$

and $\lambda_m(e) = m$

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$G(7,2)$ has 747197622 multiorientations!

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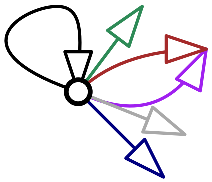
Restricting C

consider $\mathrm{Cay}_{\mathrm{col}}(S, C)$ as an arc-coloured directed multigraph

$\mathrm{Cay}(S, C)$ is $|C|$-*outregular* directed multigraph



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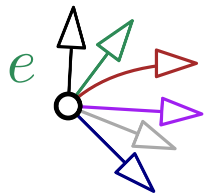

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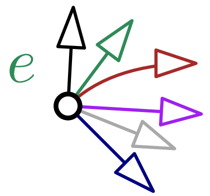

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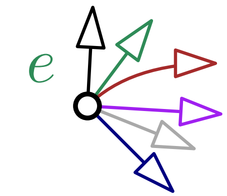

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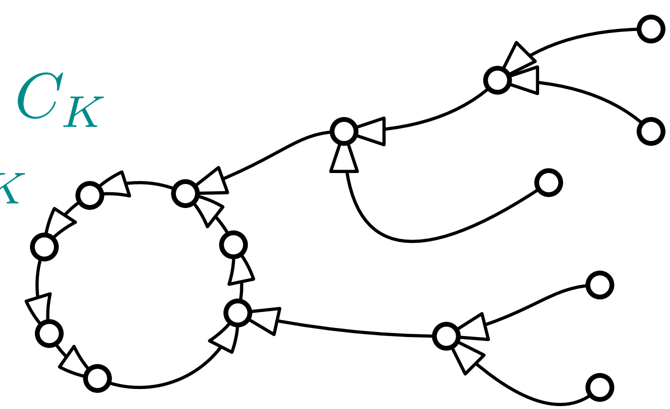

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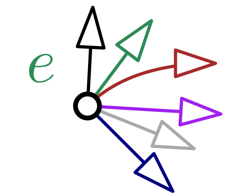

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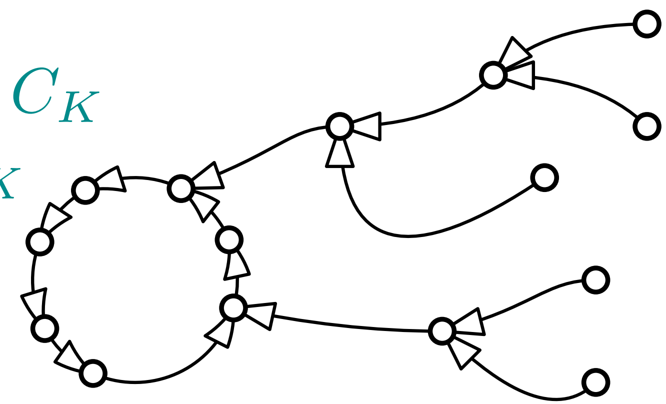

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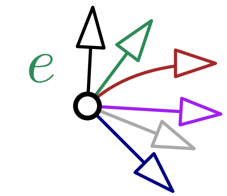

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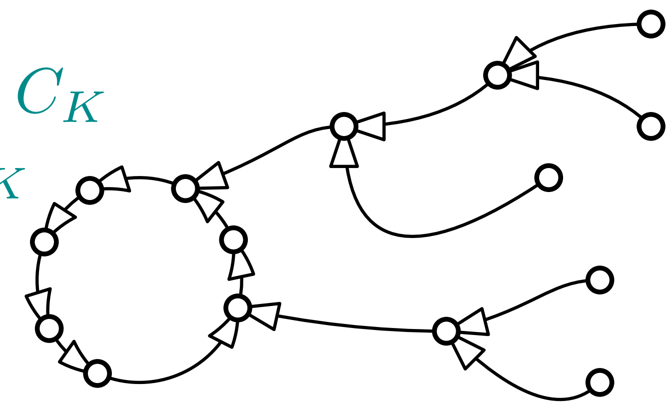

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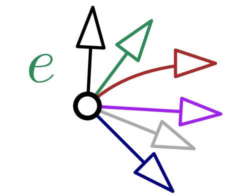

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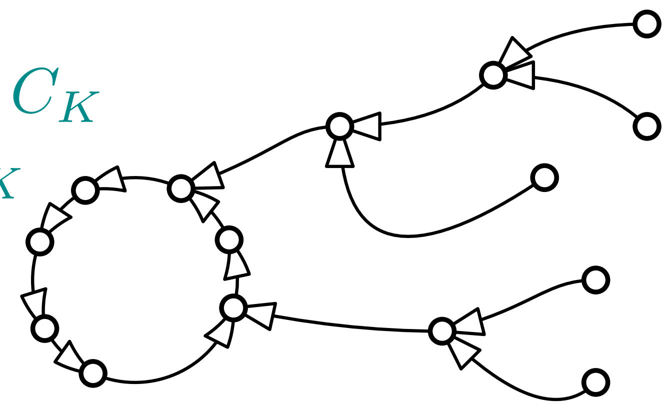

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if $< C >= M$ then everybody reachable from e

$\rightsquigarrow G(7, 2)$ still 240088032 multiorientations ($\sim 65\%$)

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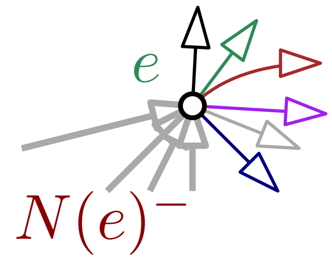

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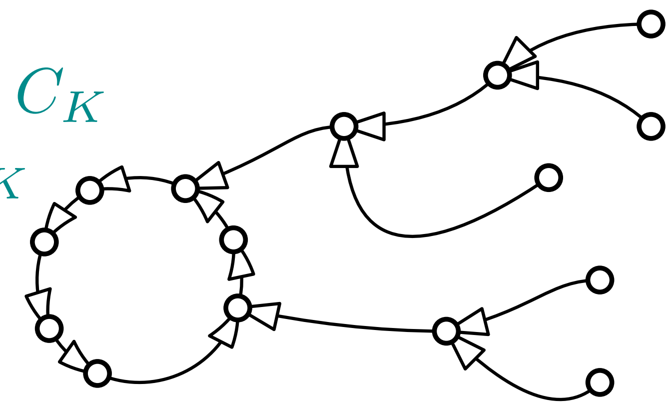

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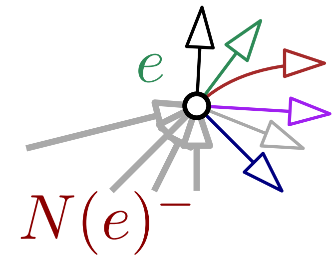

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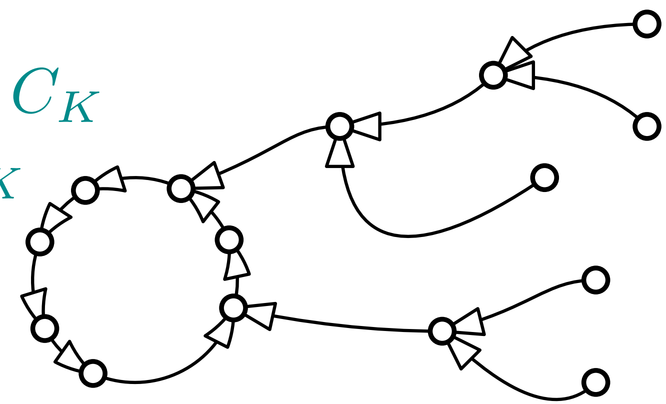

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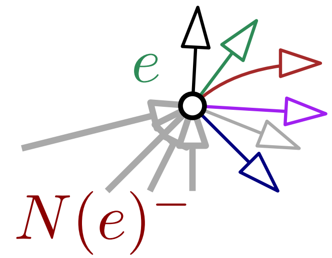

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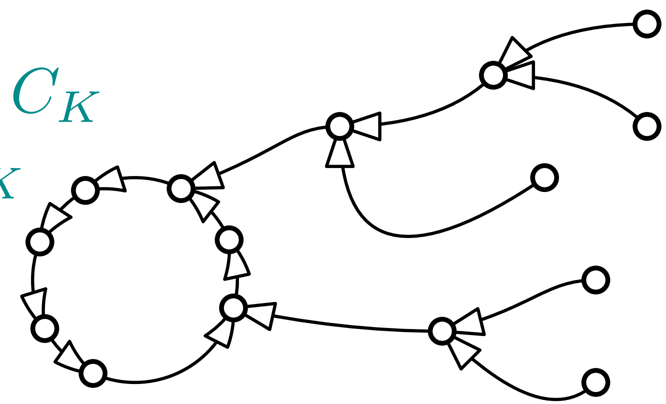

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the e-component of $\underline{\mathrm{Cay}}(M, N^{-}(e))$ is a group Cayley graph for some $\Gamma < \mathrm{Aut}(G)$

$|C| = 2$

Thm [García-Marco, K '24]:
$G(n,k) = \underline{\mathrm{Cay}}(M, C)$ with $M = \langle C \rangle$ and $|C| = 2$ if and only if:
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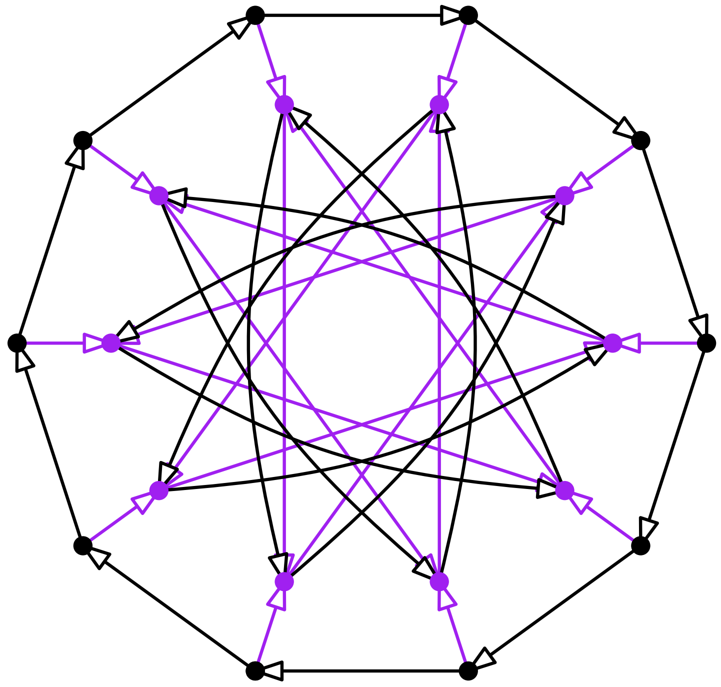

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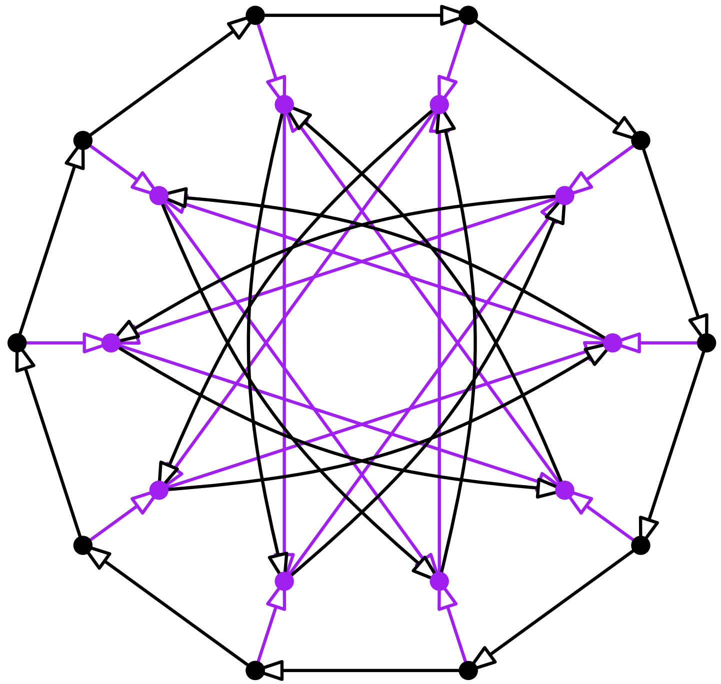

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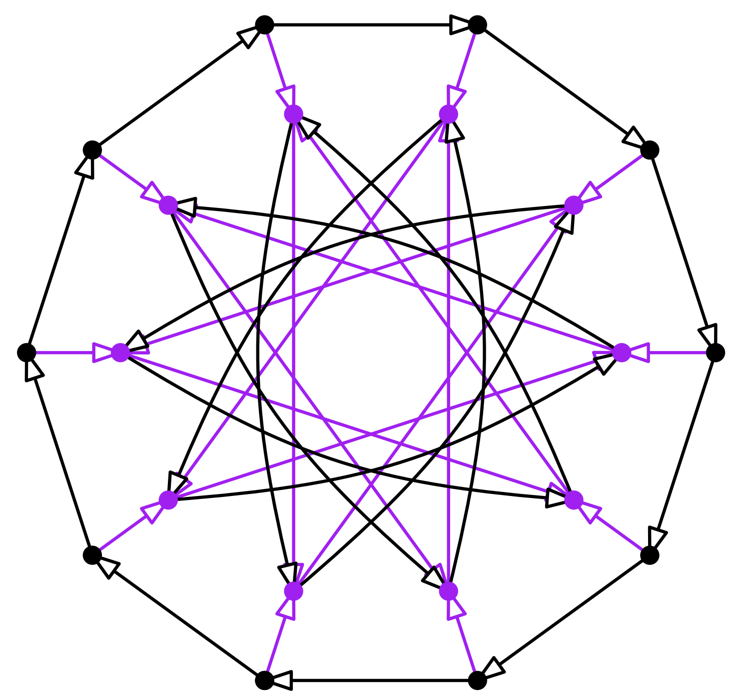

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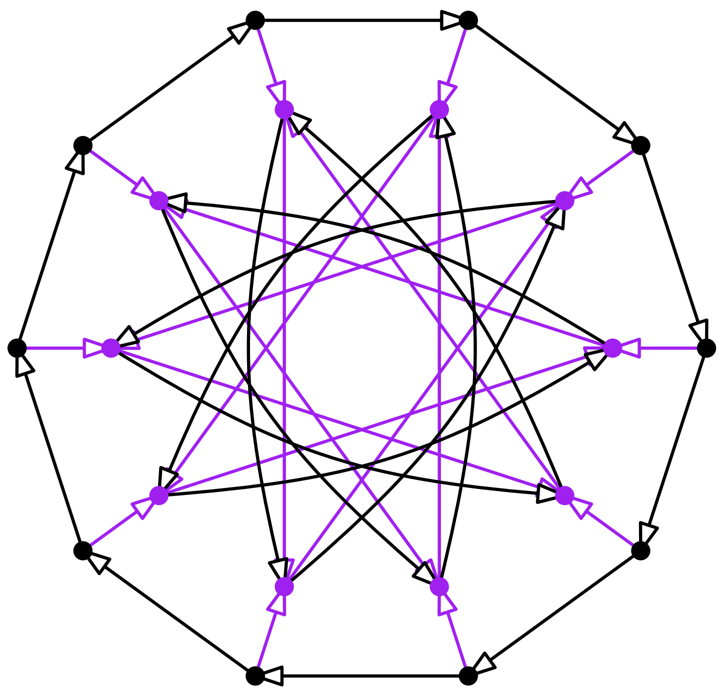

still quite some work

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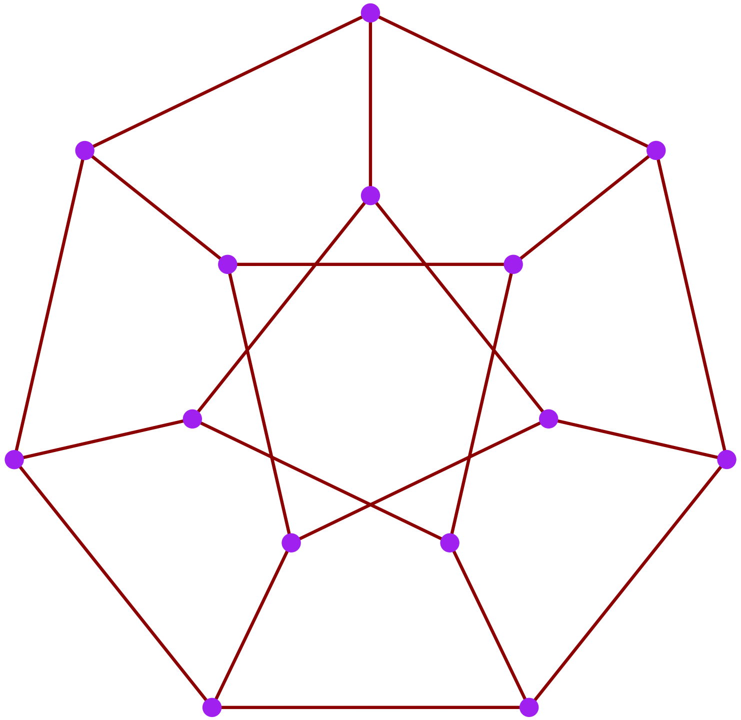
use of computer$\rightsquigarrow$

Proposition [K, Vidal '24]:
$G(7,2) = \underline{\mathrm{Cay}}(M,C) \implies C$ contains no invertible element

Open graphs

$G(7, 2)$ and many generalized Petersen graphs



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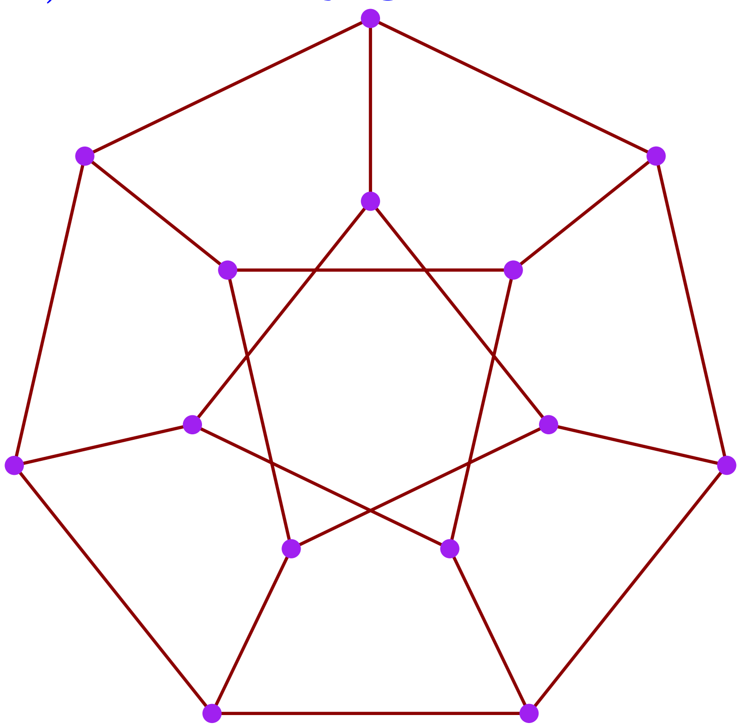
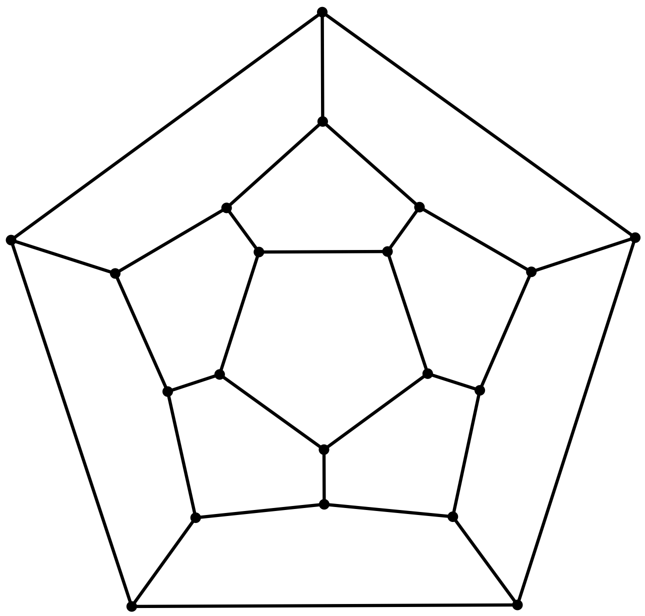


icosidodecahedron



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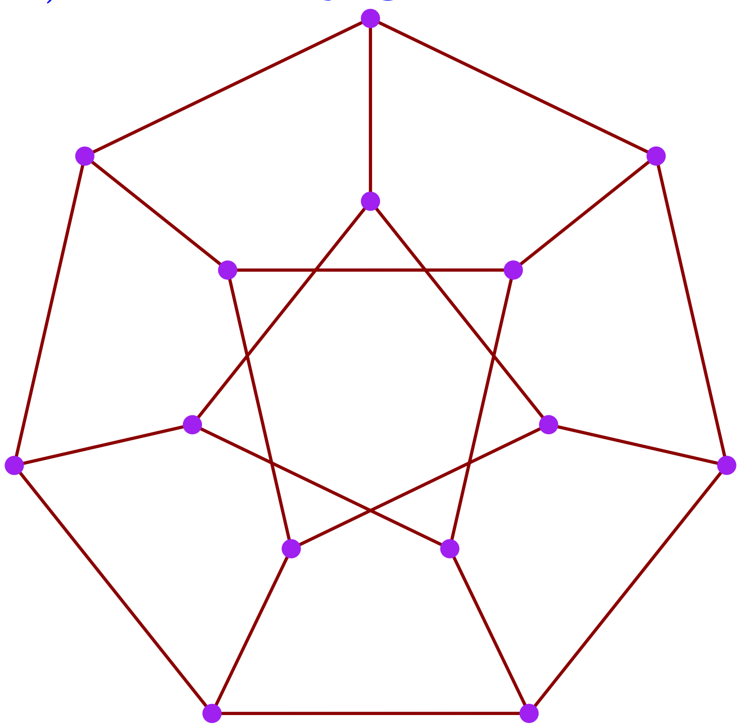

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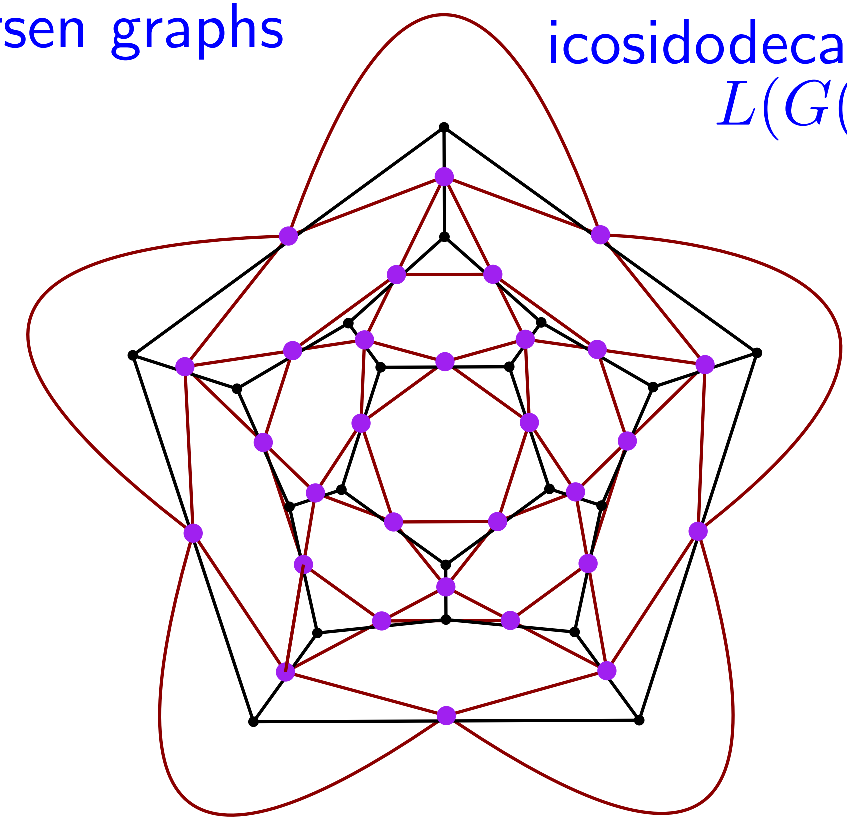

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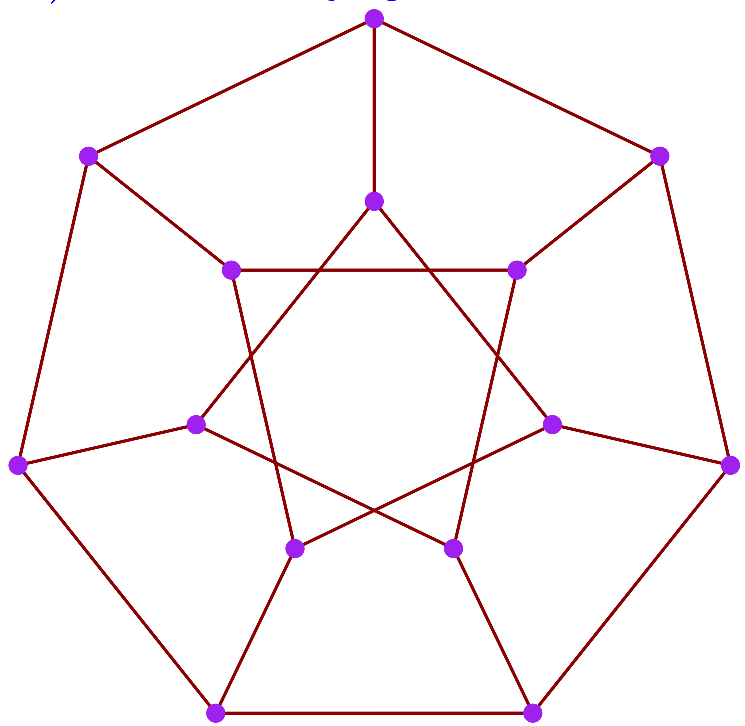

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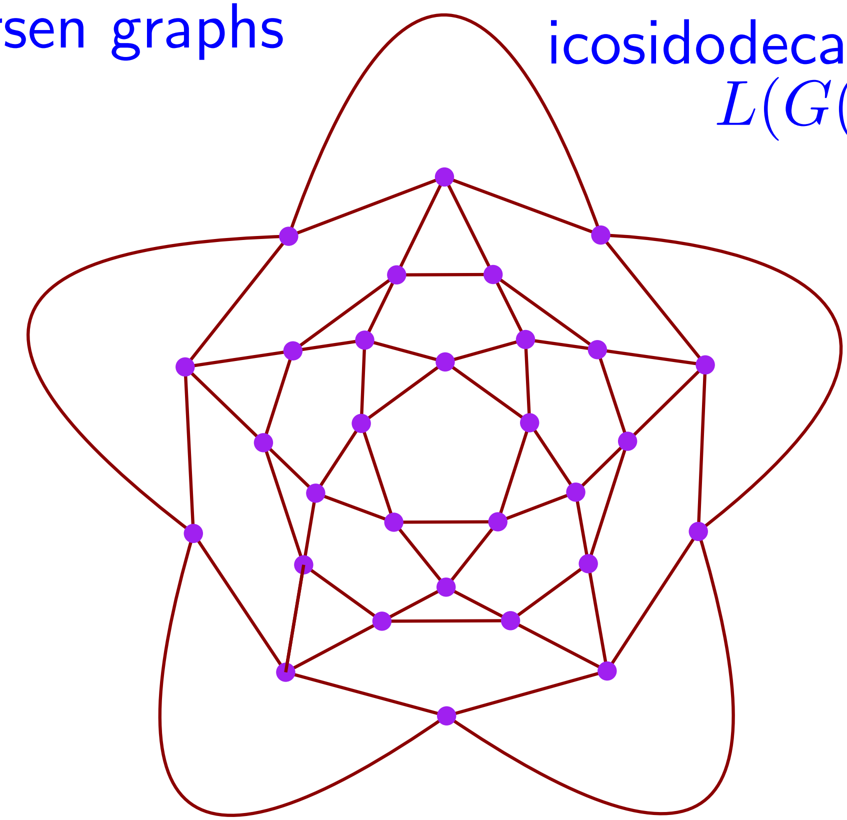

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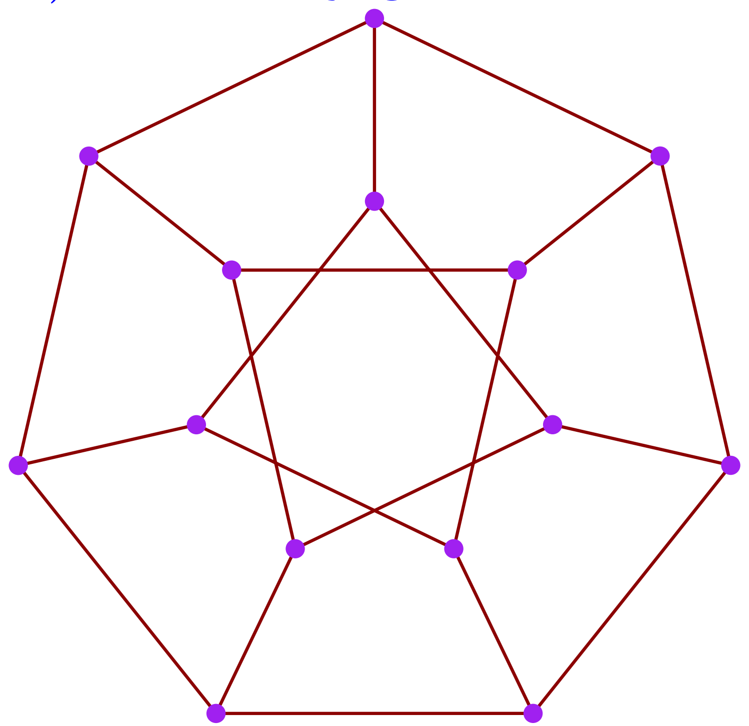

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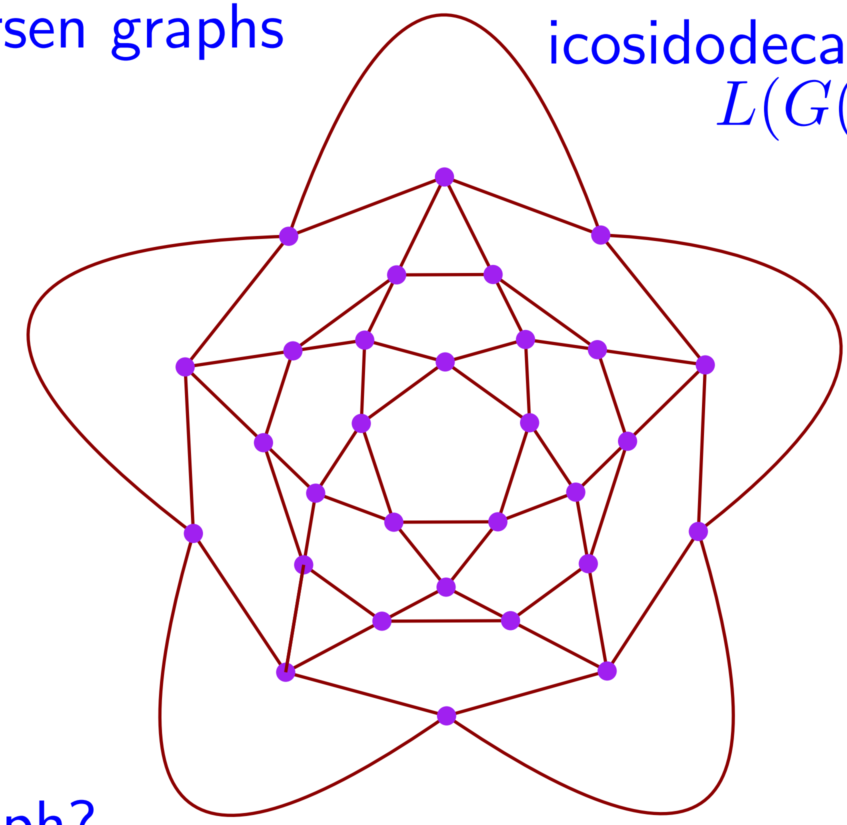

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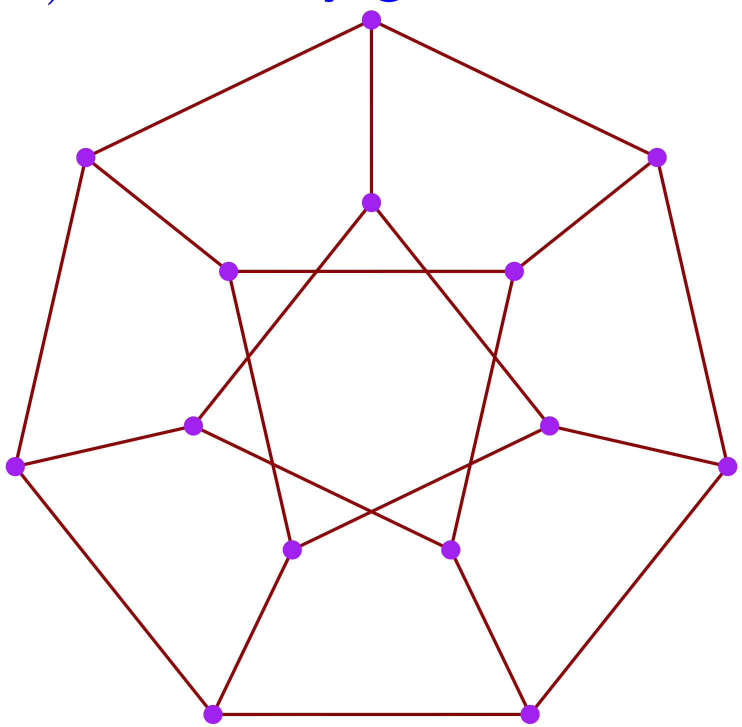

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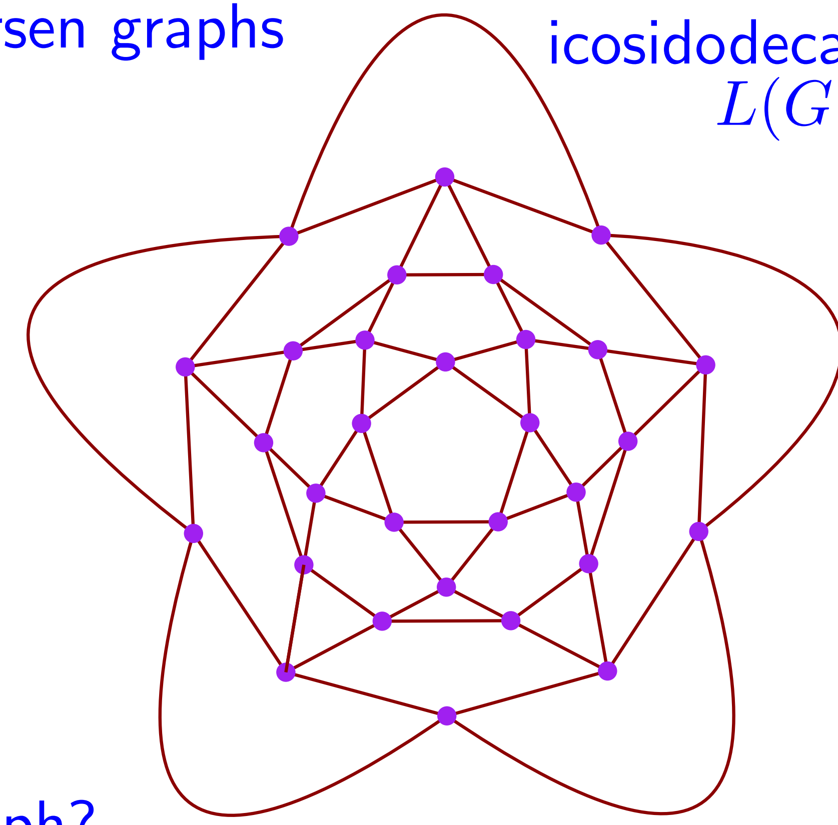

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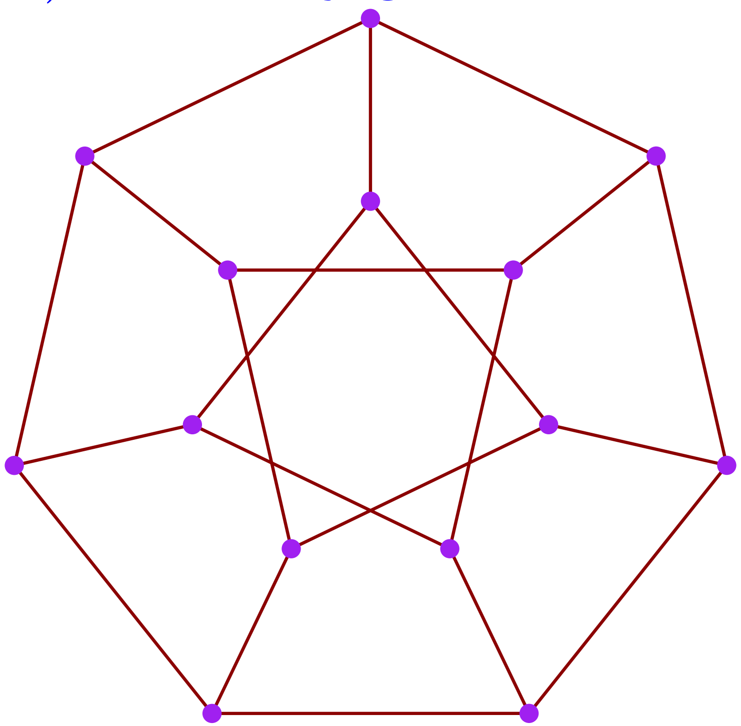

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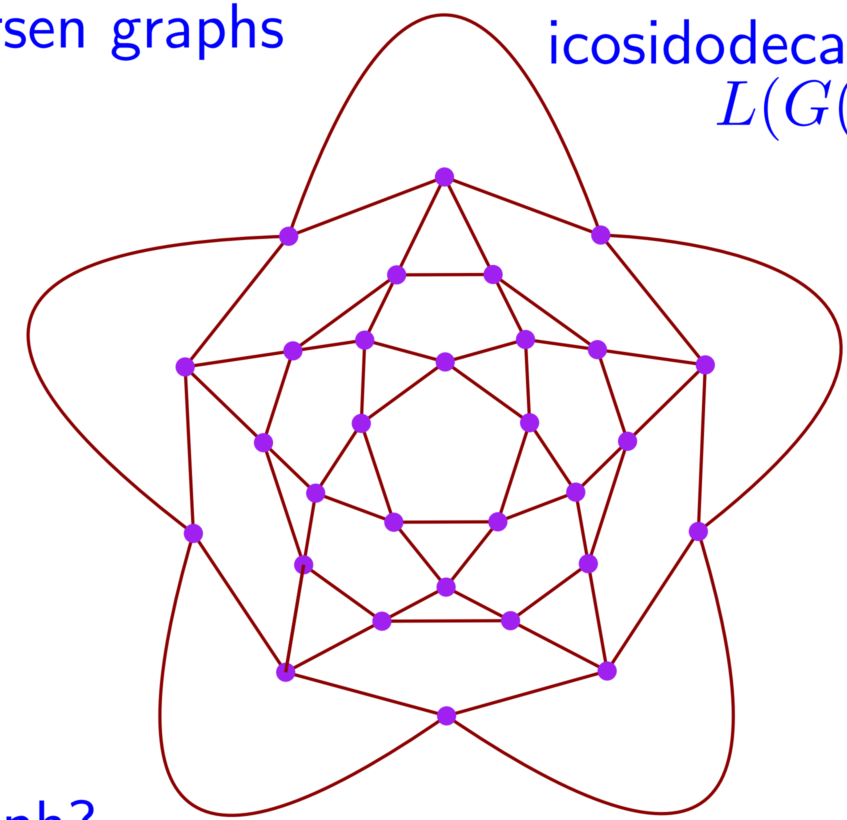

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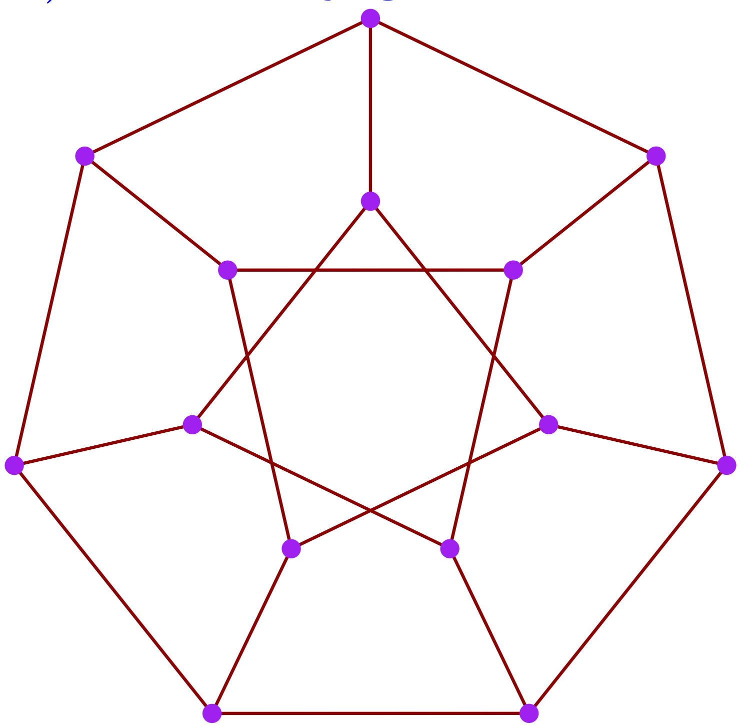

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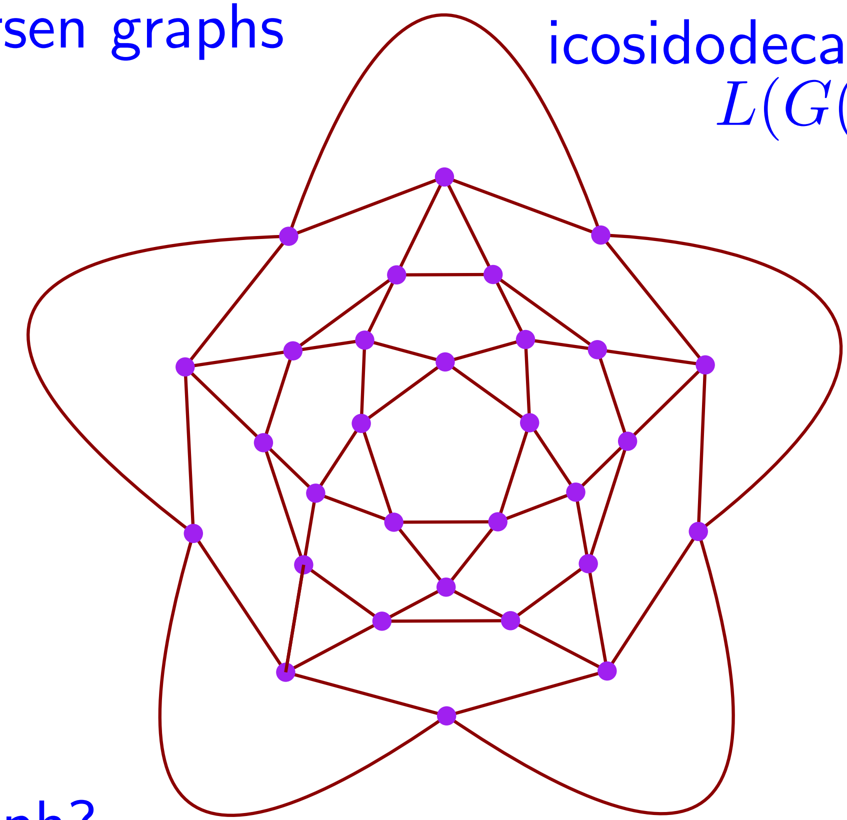

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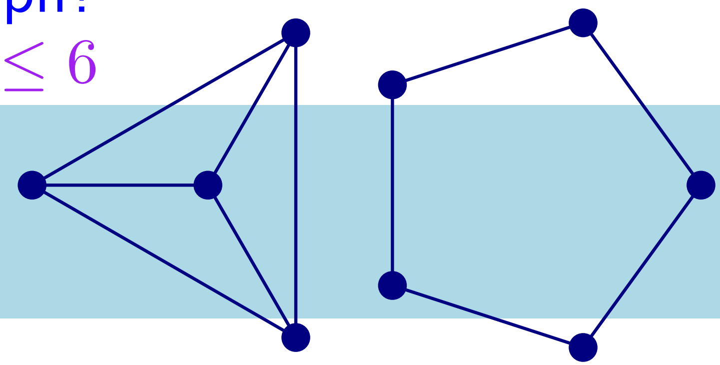

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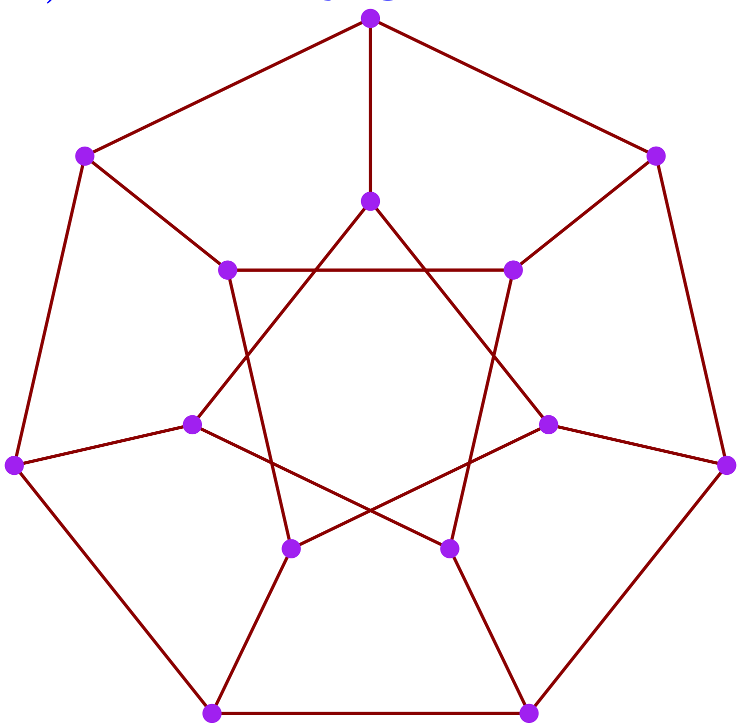

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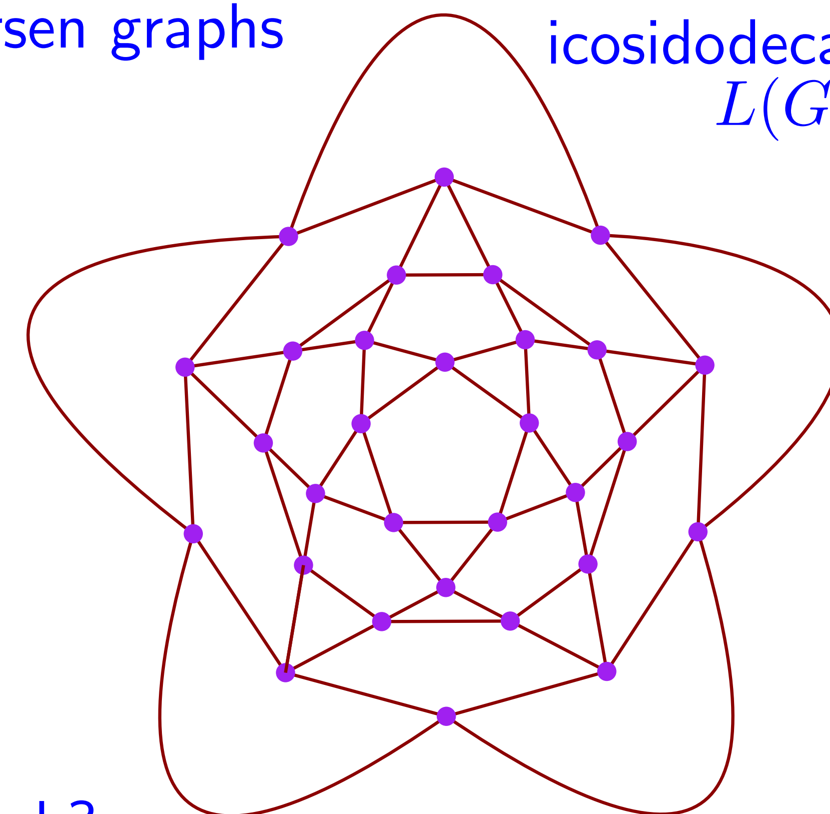

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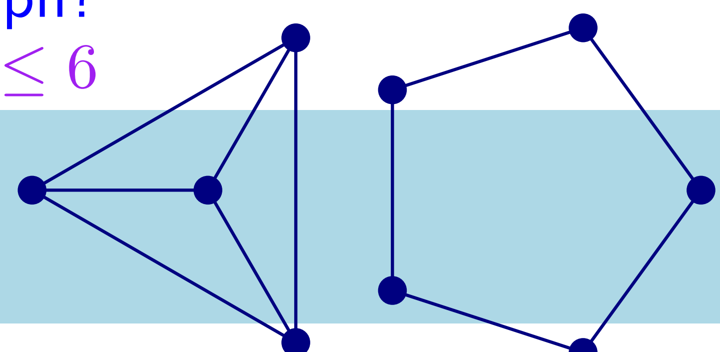

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disconnected, planar, treewidth 3

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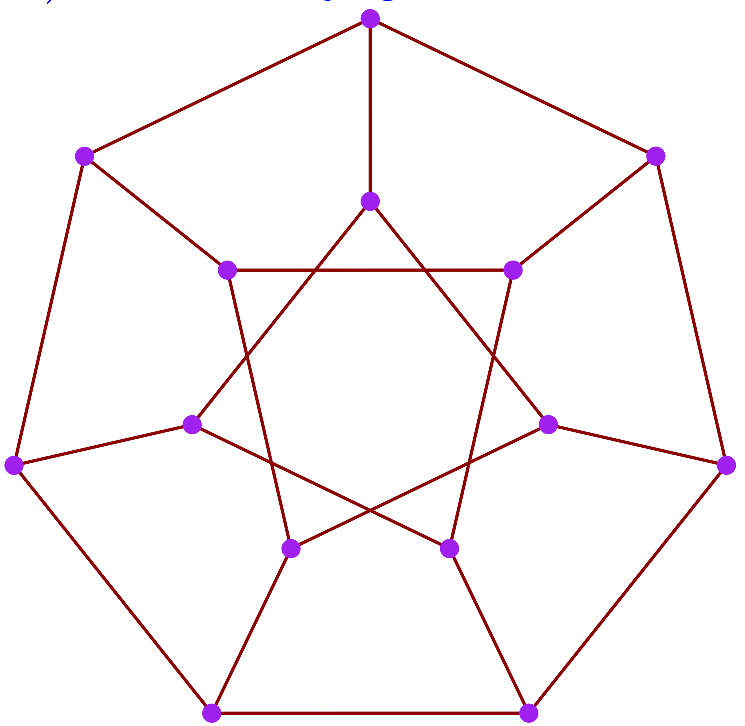

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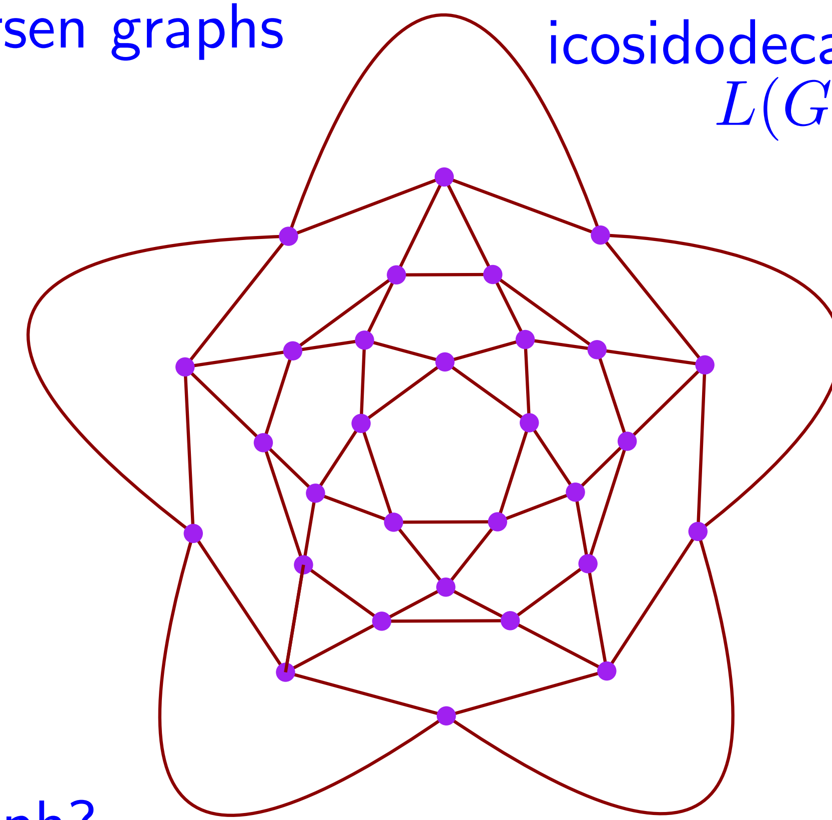

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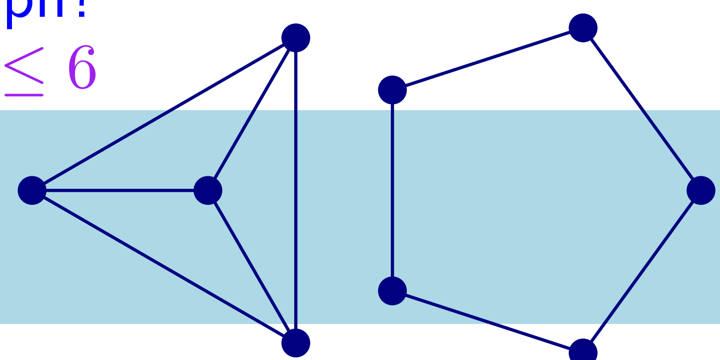

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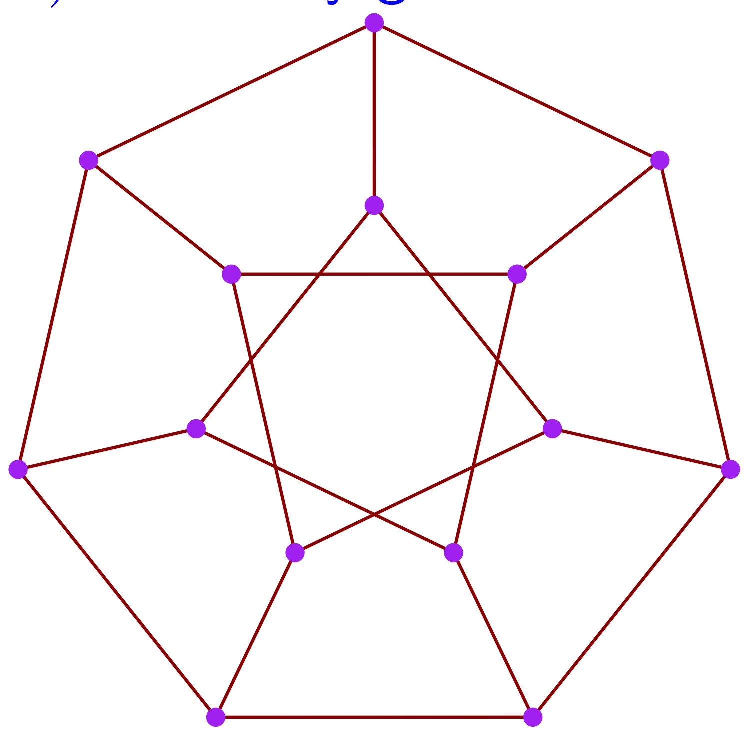

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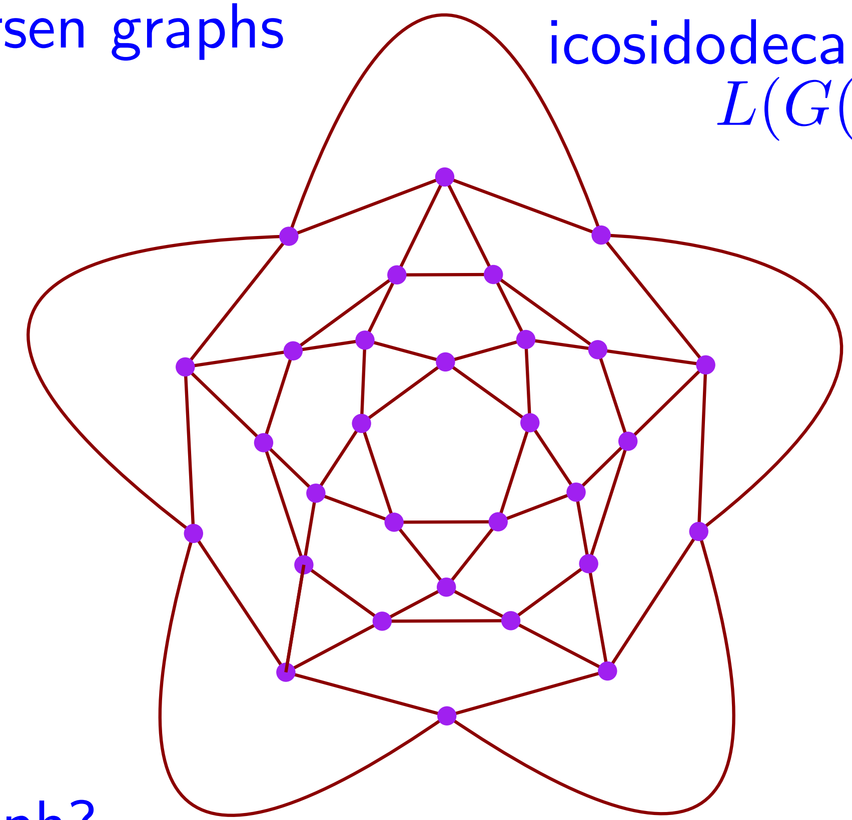

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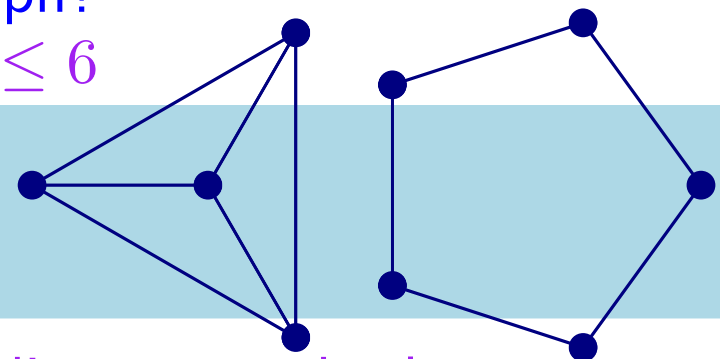

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Quest[K, Puig i Surroca '23]: is every outerplanar graph monoid graph?

Open graphs

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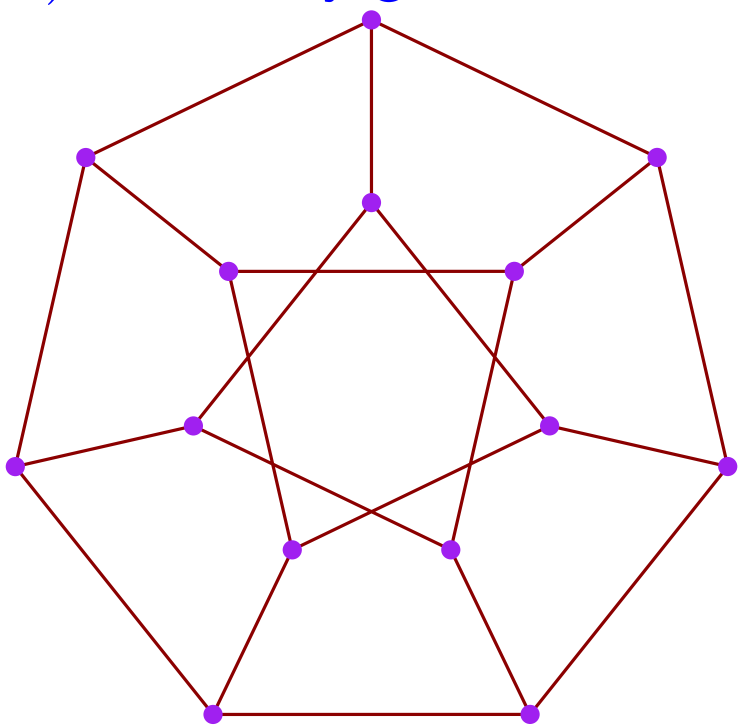

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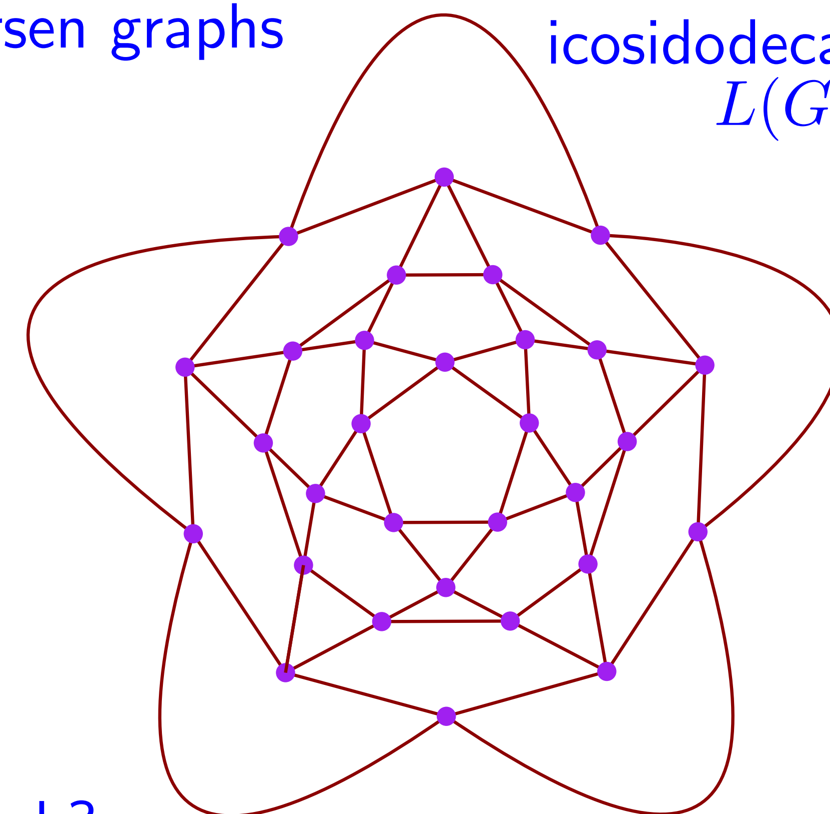

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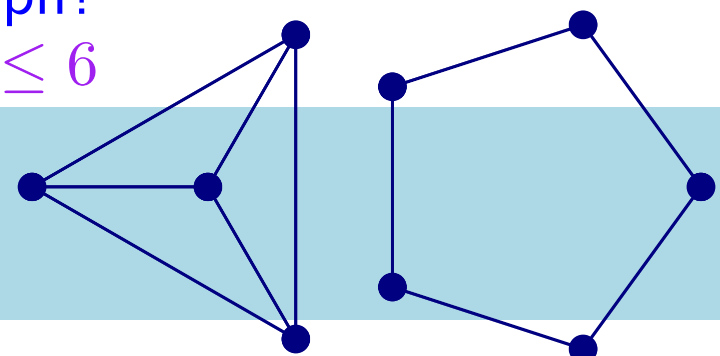

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Quest[K, Puig i Surroca '23]: is every outerplanar graph monoid graph?
is every graph a semigroup graph?

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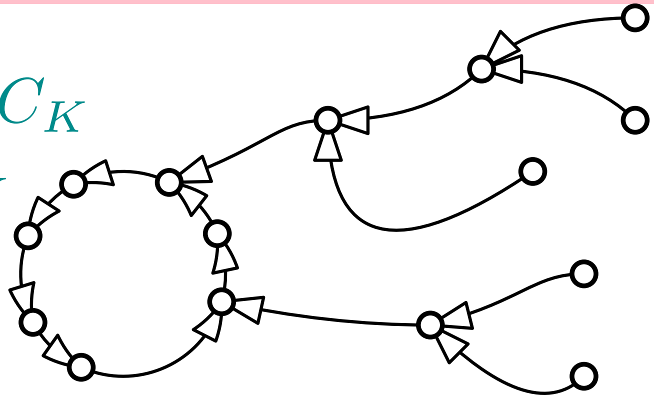

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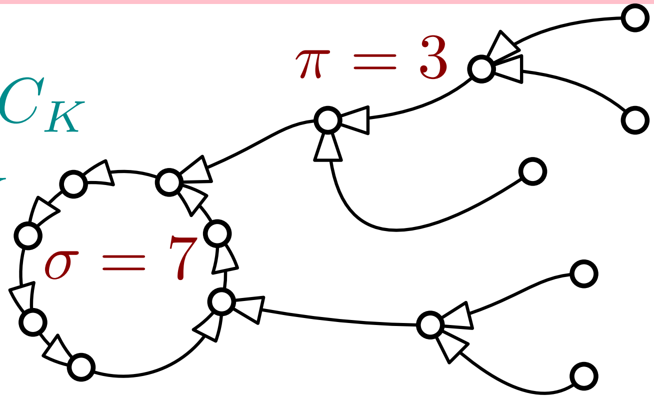

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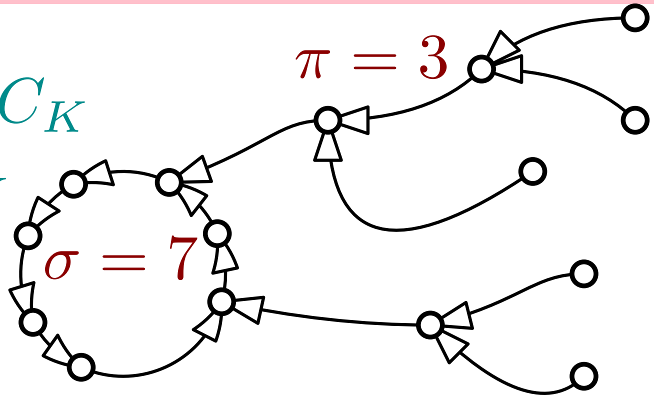

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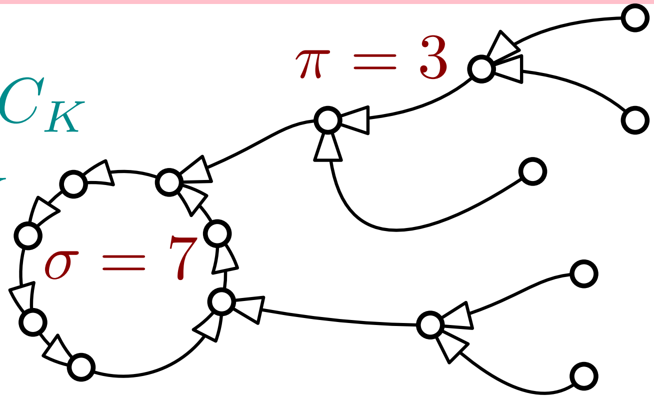

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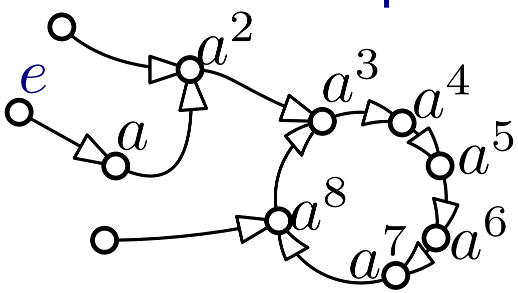
"$\Rightarrow$:" a of *index* i and *period* $p \rightsquigarrow a^i = a^{i+p}$ $\quad$ K component of e



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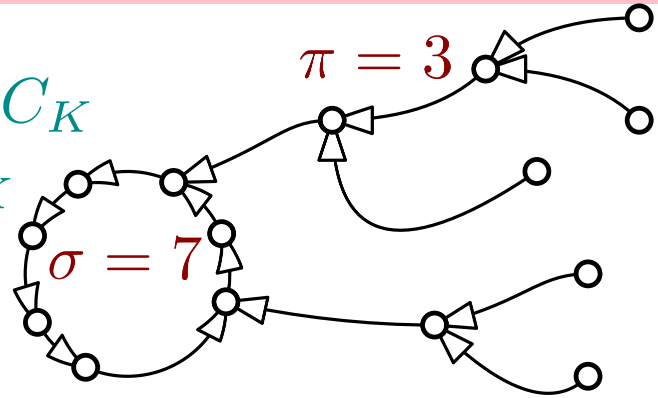

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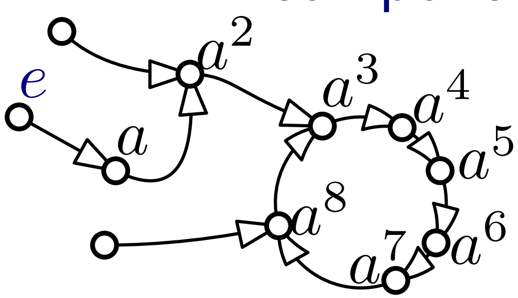

$\sigma(K) = p$
$\pi(K) \geq i$

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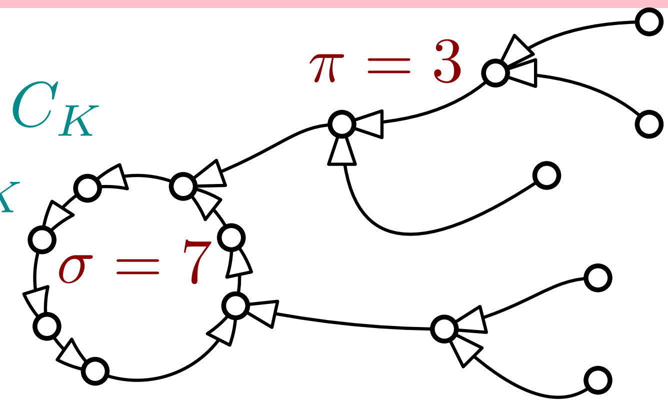

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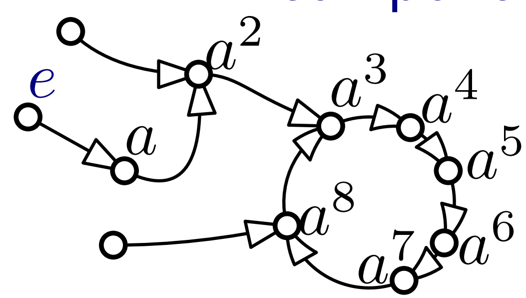

$\sigma(K) = p$
$\pi(K) \geq i$

suppose $\pi(K') > i$

$d(x, C_{K'}) > i \Rightarrow xa^i \neq xa^{i+k}$ for all $k \geq 1$

$\Rightarrow a^i \neq a^{i+p}$

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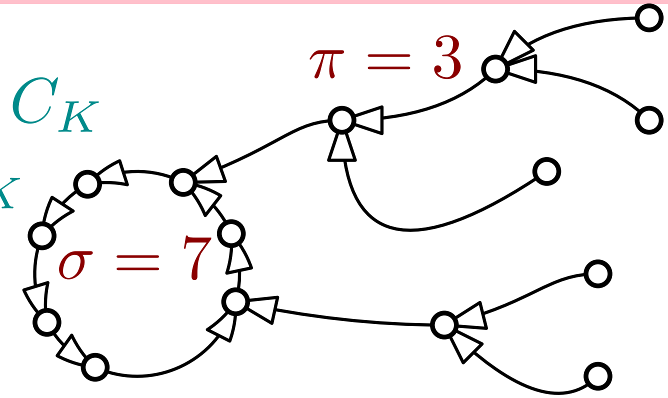

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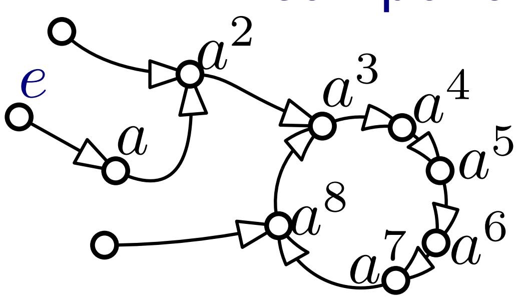
"$\Rightarrow$:" a of *index* i and *period* $p \rightsquigarrow a^i = a^{i+p}$

K component of e

suppose $\quad \pi(K') > i$

$d(x, C_{K'}) > i \Rightarrow xa^i \neq xa^{i+k}$ for all $k \geq 1$

$\Rightarrow a^i \neq a^{i+p}$



$\sigma(K) = p$

$\pi(K) \geq i$

$\pi(K') \leq i \leq \pi(K) = i$

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$D = (V, A)$ 1-*outregular* $\iff$

- each component K has unique (directed) cycle C_K
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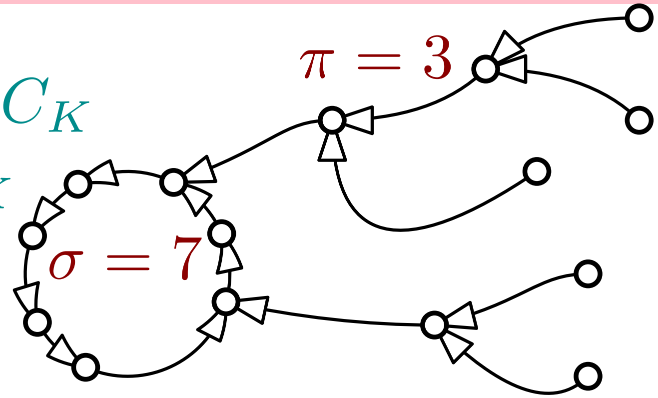

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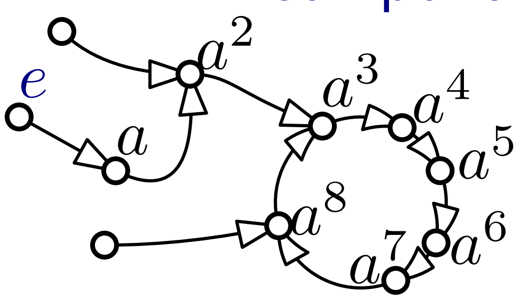

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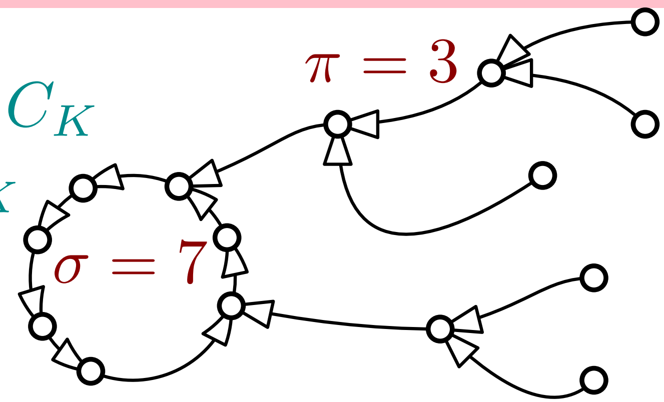

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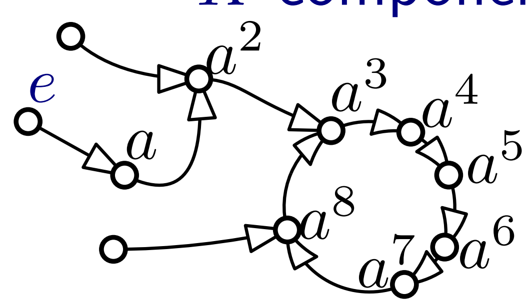

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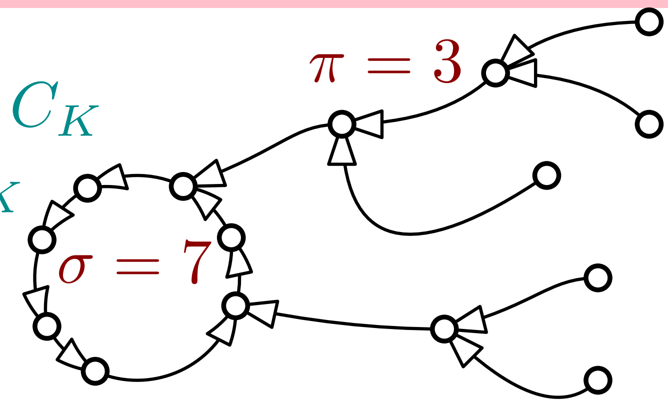

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$\rightsquigarrow$ for every pseudotree P there is monoid M with $P = \underline{\mathrm{Cay}}(M, \{a\})$

$\rightsquigarrow$ for every forest F there is monoid M with $F = \underline{\mathrm{Cay}}(M, \{a\})$