Parameterized Aspects of Distinct Kemeny Rank Aggregation

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Fixed Parameter Tractability

A problem is fixed parameter tractable or FPT with respect to parameter k if it admits an algorithm with runtime $f(k) \cdot n^{\mathcal{O}(1)}$

Candidates and Votes

We have a set of votes Π and a set of candidates C. Every vote $\pi \in \Pi$ is assumed to be a linear order over C. Let $|C| = m$ and $|\Pi| = n$.

Figure 1: Votes in Π where $C = \{a, b, c, d, e\}$

$$\pi_1 \quad b \succ d \succ c \succ a \succ e$$
$$\pi_2 \quad d \succ a \succ c \succ b \succ e$$
$$\pi_3 \quad a \succ b \succ c \succ d \succ e$$
$$\vdots$$

Distance between any two votes

Intuitively the Kendall-Tau distance (*KT-Dist* in short) is the number of pairwise disagreements between two votes $\pi_1, \pi_2 \in \Pi; \pi_1 \neq \pi_2$. Formally $\text{KT-Dist}(\pi_1, \pi_2) := |\{(c, c') \in C \times C \mid c <_{\pi_1} c' \wedge c' <_{\pi_2} c\}|$

Figure 2: Between two votes KT-Dist = # pairs colored red

$$\pi_1 \; b \succ d \succ c \succ a \succ e$$
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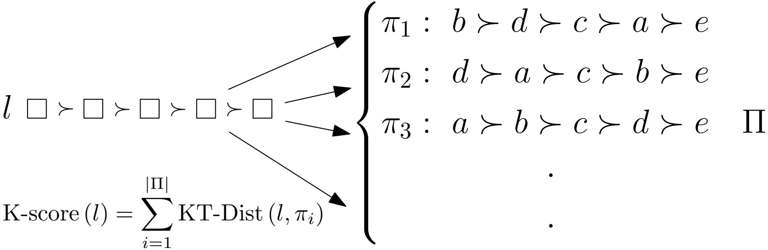
Kemeny Consensus

Figure 3: *l* is the Kemeny Consensus if K-score(l) is minimum



$$\text{K-score}\,(l) = \sum_{i=1}^{|\Pi|} \text{KT-Dist}\,(l, \pi_i) \leq n\binom{m}{2}$$

Finding optimal Kemeny ranking is NP-complete[1]

[1] Dwork et al.[1] showed that the problem is NP-complete even when restricted to instances with only four votes.

Preliminary

- $\text{pos}_\pi(c) := |\{c' \in C : c' \succ_\pi c\}|$
- *range* $rng(c) := \max_{\pi_i,\pi_j \in \Pi} \left\{ |\text{pos}_{\pi_i}(c) - \text{pos}_{\pi_j}(c)| \right\} + 1$ in Π.
- $\rho = \{(a, b) : a, b \in C, \forall \pi \in \Pi \text{ prefers } a \succ b\}$ i.e. unanimity order with respect to Π
- $K(\Pi) :=$ Set of (optimal) Kemeny rankings with respect to Π
- $K(\Pi, k) :=$ Set of Kemeny rankings with Kemeny score at most some integer k with respect to Π
- $N_\Pi(x \succ y) := \#$linear orders in Π where x is preferred over y.

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Problem Definition

Distinct Kemeny Ranking Aggregation (Dist. KRA)

Input : Π, C, k, r

Compute : $\ell = \min\{r, |K(\Pi, k)|\}$ distinct rankings such that $\forall i \in [\ell]$ π_i respects unanimity order of Π and K-score$(\pi_i) \leq k$ for all $i \in [\ell]$.

Arbitrary Instance : (C, Π, k, r)

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Difference between Dist. OPT KRA and Dist. Approx. KRA

In case of **Dist. OPT KRA**,

- we Compute $\ell = \min\{r, |K(\Pi)|\}$ distinct Kemeny rankings

and in case of **Dist. Approx. KRA**,

- we compute $\ell = \min\{r, |K(\Pi, \lambda \cdot k_{OPT}(\Pi))|\}$ distinct rankings such that $\forall i \in [\ell]$ π_i respects ρ and K-score$(\pi_i)_{i \in [\ell]} \leqslant \lambda \cdot k_{OPT}(\Pi)$

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Turing Reduction from Dist. OPT KRA to Dist. KRA

- If there exists an algorithm for Dist. KRA running in time $\mathcal{O}(f(m, n))$, then there exists an algorithm for Dist. OPT KRA running in time $\mathcal{O}(f(m, n)\log(mn))$
- Optimal Kemeny score $\in \{0, 1, \ldots, n\binom{m}{2}\}$
- Perform binary search in the range from 0 to $n\binom{m}{2}$ to find smallest k s.t. the algorithm for Dist. KRA returns at least one ranking.

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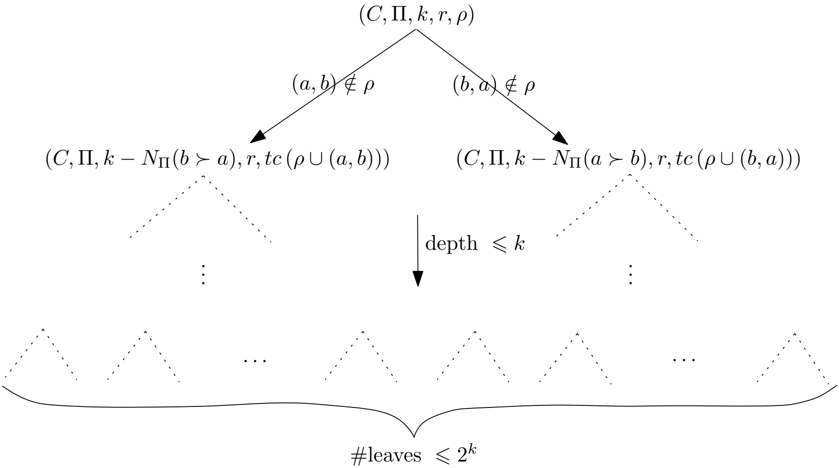
Summary of Results

Param.	Dist. OPT KRA	Dist. KRA	Dist. Approx. KRA
k	$\mathcal{O}^*\left(2^k\right)$		$\mathcal{O}^*\left(2^{\lambda k}\right)$
$\lvert C \rvert = m$	$\mathcal{O}^*\left(2^m r^{\mathcal{O}(1)}\right)$		$\mathcal{O}^*\left(2^m r^{\mathcal{O}(1)}\right)$
d	$\mathcal{O}^*\left(16^d\right)$		$\mathcal{O}^*\left(16^{\lambda d}\right)$
$r_{\max}$	$\mathcal{O}^*\left(32^{r_{\max}}\right)$		
$w,\ r$	$\mathcal{O}^*\left(2^{\mathcal{O}(w)} \cdot r\right)$		$\mathcal{O}^*\left(2^{\mathcal{O}(w)} \cdot r\right)$

Table 9.1: Summary of Results.

DIST. KRA parameterized by k



Runtime : $\mathcal{O}^\star\left(2^k\right)$

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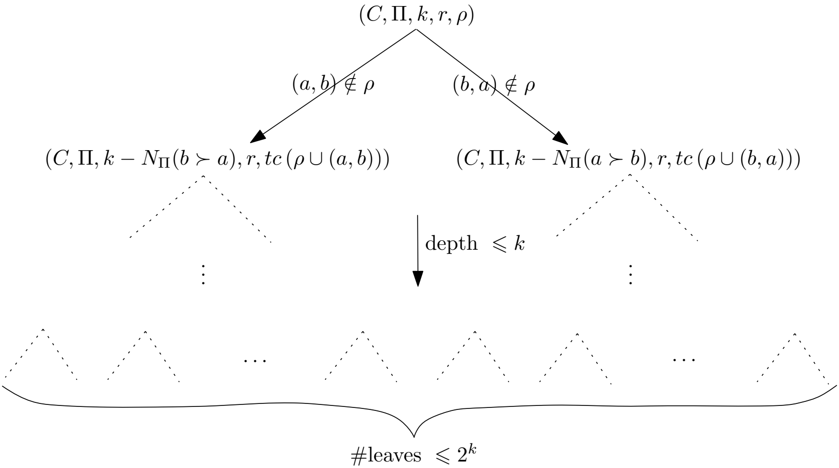

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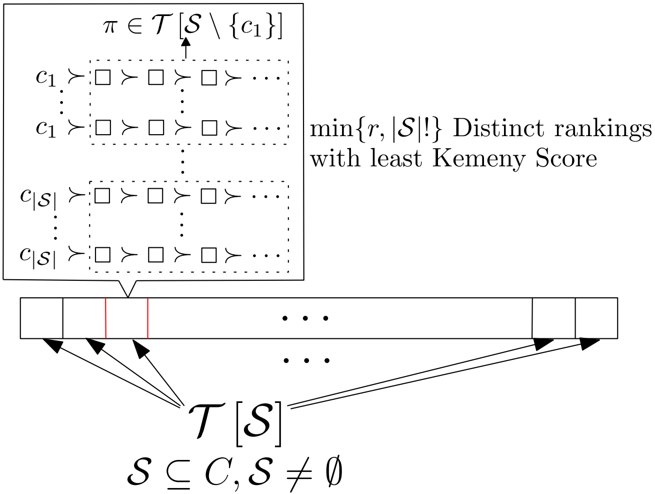
Dist. Approx. KRA parameterized by k and λ

If we replace k with λk in our previous algorithm, we come up with the following :-
There is an algorithm for Distinct Approximate Kemeny Ranking Aggregation running in time $\mathcal{O}^*\left(2^{\lambda k}\right)$ parameterized by both λ and k.

DIST. KRA parameterized by $|C| = m$



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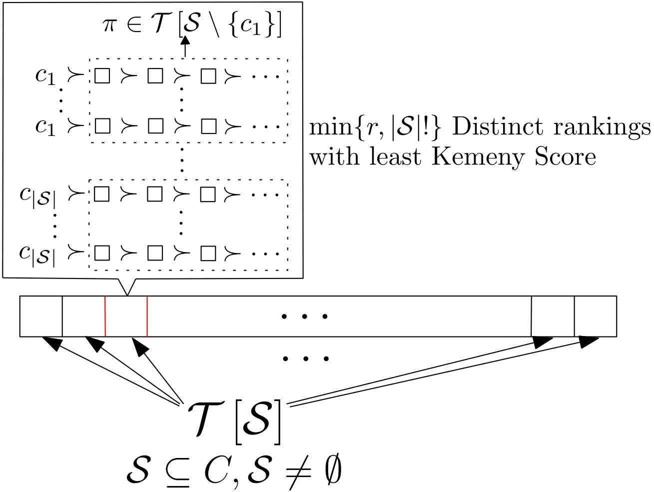

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A short proof of correctness

- Let $\mathcal{S} = \{c_1, c_2, \ldots, c_\ell\}$ and let $c_1 \succ c_2 \succ \cdots \succ c_\ell$ be one of the Kemeny rankings stored in $\mathcal{T}[\mathcal{S}]$.
- Then $c_1 \succ c_2 \succ \cdots \succ c_\ell$ is a Kemeny Ranking when votes of Π are restricted to $\mathcal{S}$.
- But then $c_2 \succ \cdots \succ c_\ell$ is also a Kemeny ranking when votes of Π are restricted to $\mathcal{S} \setminus \{c_1\}$.
- If not then K-score$(c'_2 \succ \cdots \succ c'_\ell) <$ K-score$(c_2 \succ \cdots \succ c_\ell)$
- Consequently K-score$(c_1 \succ c'_2 \succ \cdots \succ c'_\ell) <$ K-score$(c_1 \succ c_2 \succ \cdots \succ c_\ell)$
- Contradicting our assumption that $c_1 \succ c_2 \succ \cdots \succ c_\ell$ is a Kemeny ranking

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Dist. Approx. KRA parameterized by m

1. Let (C, Π, λ, r) be an instance of Dist. Approx. KRA.
2. Run the previous algorithm on instances $(C, \Pi, 0, 1), (C, \Pi, 1, 1), \ldots$ of Dist. KRA.
3. Stop once we encounter a YES instance, say $(C, \Pi, k^*, 1)$ where k^* is the optimum Kemeny score.
4. Next, run the same algorithm on the instance $(C, \Pi, \lambda \cdot k^*, r)$ of Dist. KRA. As $k^* \leq \binom{m}{2} \cdot |\Pi|$, the overall running time $\leq \mathcal{O}^*(2^m r^{\mathcal{O}(1)})$.
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DIST. OPT KRA parameterized by d

- $p_{avg}(c) \coloneqq \frac{1}{n} \cdot \sum_{\pi \in \Pi} \text{pos}_\pi(c)$.
- Formally for an election (Π, C), $d \coloneqq \frac{\sum_{v \in \Pi} \sum_{w \in \Pi} \text{KT-Dist}(v,w)}{n \cdot (n-1)}$.
- Betzler et al. [2][2]. showed that $p_{avg}(c) - d < \text{pos}_\pi(c) < p_{avg}(c) + d$ where π is an optimal Kemeny ranking.
- $P_i \coloneqq \{c \in C \mid p_{avg}(c) - d < i < p_{avg}(c) + d\} \; \forall i \in [m-1]_0$ in an optimal Kemeny Consensus and $|P_i| \leqslant 4d \;\; \forall i \in [m-1]_0$; [2]
- $$\text{pK-score}(c, A) \coloneqq \sum_{c' \in A} \sum_{\pi \in \Pi} d_\pi^A(c, c') \Big[\text{where } d_\pi^A(c,c') \coloneqq 0 \text{ if } c >_\pi c' \text{ and } d_\pi^A(c,c') \coloneqq 1 \text{ otherwise}\Big]$$

[2] [2] Nadja Betzler, Michael R. Fellows, Jiong Guo, Rolf Niedermeier, and Frances A. Rosamond. "Fixed-parameter algorithms for kemeny rankings." Theor. Comput. Sci., 410(45):4554-4570,2009.

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[2] [2] Nadja Betzler, Michael R. Fellows, Jiong Guo, Rolf Niedermeier, and Frances A. Rosamond. "Fixed-parameter algorithms for kemeny rankings." Theor. Comput. Sci., 410(45):4554-4570,2009.

DIST. OPT KRA parameterized by d

- $p_{avg}(c) := \frac{1}{n} \cdot \sum_{\pi \in \Pi} \text{pos}_\pi(c)$.
- Formally for an election (Π, C), $d := \frac{\sum_{v \in \Pi} \sum_{w \in \Pi} \text{KT-Dist}(v,w)}{n \cdot (n-1)}$.
- Betzler et al. [2][2]. showed that $p_{avg}(c) - d < \text{pos}_\pi(c) < p_{avg}(c) + d$ where π is an optimal Kemeny ranking.
- $P_i := \{c \in C \mid p_{avg}(c) - d < i < p_{avg}(c) + d\} \, \forall i \in [m-1]_0$ in an optimal Kemeny Consensus and $|P_i| \leqslant 4d \quad \forall i \in [m-1]_0$; [2]
-

$$\text{pK-score}(c, A) := \sum_{c' \in A} \sum_{\pi \in \Pi} d_\pi^A(c, c') \Big[\text{where } d_\pi^A(c, c') := 0 \text{ if } c >_\pi c' \text{ and } d_\pi^A(c, c') := 1 \text{ otherwise} \Big]$$

2 [2] Nadja Betzler, Michael R. Fellows, Jiong Guo, Rolf Niedermeier, and Frances A. Rosamond. "Fixed-parameter algorithms for kemeny rankings." Theor. Comput. Sci., 410(45):4554-4570,2009.

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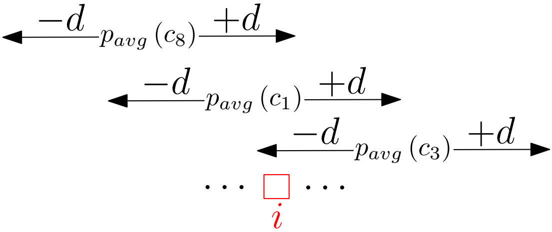
[2] [2] Nadja Betzler, Michael R. Fellows, Jiong Guo, Rolf Niedermeier, and Frances A. Rosamond. "Fixed-parameter algorithms for kemeny rankings." Theor. Comput. Sci., 410(45):4554-4570,2009.

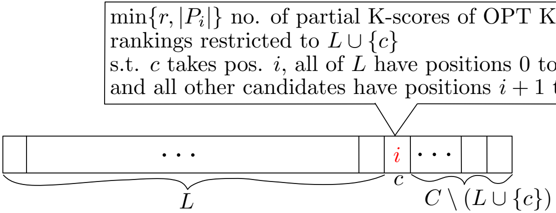
Description of P_i



$$P_i = \{c_1, c_3, c_8\}$$

Figure 4: Visual representation of P_i

FPT algorithm for Dist. OPT KRA parameterized by d



$\forall i \in [m-1], \forall c \in P_i$ and $\forall P'_i \subseteq P_i \setminus \{c\}$

$$\mathcal{T}(i, c, P'_i) = \min_{\text{all possible orderings over } L} \{\text{K-score of the partial ranking over } L \text{ when } \Pi \text{ is restricted to } L\} + \text{pK-score}(c, C \setminus (L \cup \{c\}))$$

Figure 5: DP Algorithm

Runtime: $\mathcal{O}^*\left(16^d\right)$

DIST. APPROX. KRA parameterized by both d and λ

Lemma

$p_{avg}(c) - \lambda \cdot d \leq pos_{\pi}(c) \leq p_{avg}(c) + \lambda \cdot d$

Based on the proof we can argue that $|P_i| \leq 4\lambda d$ and hence we can derive the runtime bound of the FPT algorithm $\mathcal{O}^*\left(16^{\lambda d}\right)$.

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[3] Arrighi, E., Fernau, H., de Oliveira Oliveira, M., & Wolf, P. (2020). Width notions for ordering-related problems.

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Thank You