Understanding visibility graphs of point sets

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Overview

Introduction

Recognition problem

Optimization problems

Disjoint triangle partition

Generalized cycle partition

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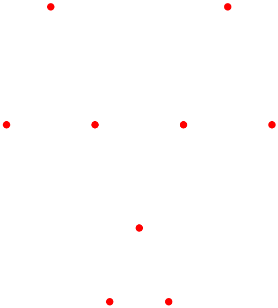
- Visibility graphs are widely studied structures in computational geometry, and may be defined on point sets, line segments, polygons and other geometric sets.

- Visibility graphs have their use in robot motion planning, security problems etc.

Introduction



$P = \{p_1, p_2, \ldots, p_n\}$ is a given set of points on the plane.

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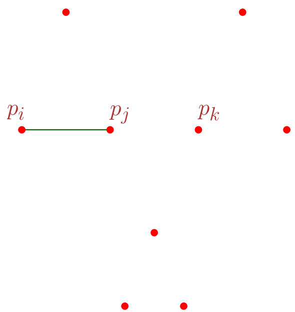

Two points p_i and p_j of P are *mutually visible* if the line segment p_ip_j does not contain or pass through any other point of P.

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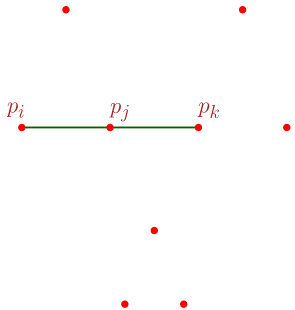

Two points p_i and p_k of P are *mutually invisible* if the line segment $p_i p_j$ contains or passes through another point of P.

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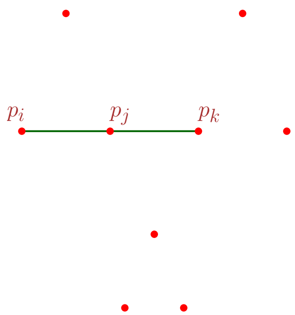

A point p_j of P lying on the line segment p_ip_k, is called a *blocker* of p_i and p_k.

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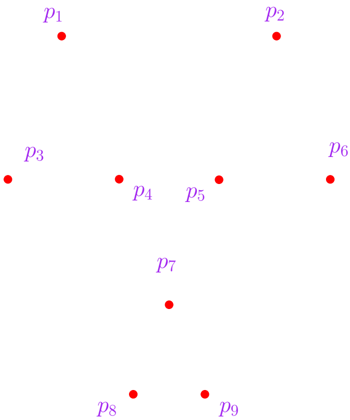

The *point visibility graph* (PVG) of P is a graph $G = (V, E)$, s.t. $v_i \in V \Leftrightarrow p_i \in P$, **and** $v_i v_j \in E \Leftrightarrow p_i$ *and* p_j **are mutually visible** (Ghosh, 2007, Ghosh et al, 2010).

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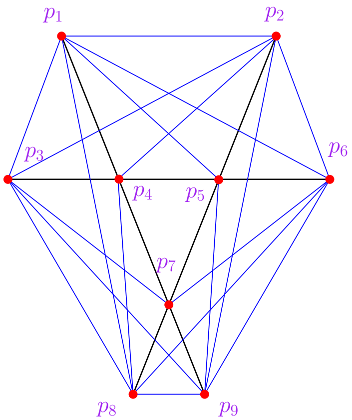

$G(V, E)$:

$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$

$E = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5),$
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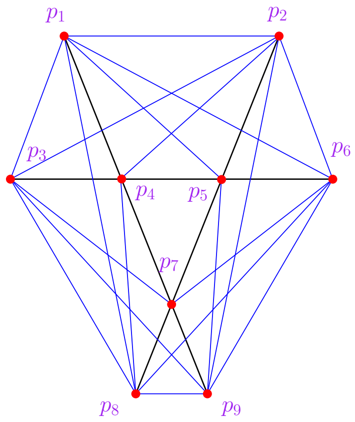

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Given a point set P in the plane, its *PVG* can be computed in $O(n^2)$ time by using the results of Chazelle et al. (1985) or Edelsbrunner et al. (1986).

Recognition problem

Given a point set P in the plane, its PVG can be computed in polynomial time.

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Problem definition:

Given a graph G, is there a point set P on the plane such that G is the PVG of P?

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Problem definition:

Given a graph G, is there a point set P on the plane such that G is the *PVG* of P?

The above problem is called the *recognition* problem for *PVGs*. Such a point set, if it exists, is called a *visibility embedding* of G.

Recognition problem

Results:

- Ghosh and Roy (2015) provided the following.

  (a) Three necessary conditions for recognizing PVGs,

  (b) They showed that the recognition problem for PVGs is in PSPACE,

  (c) They gave a complete characterization, and using the characterzation they designed a linear time algorithm for the recognition of planar PVGs.

Recognition problem

Results:

- Roy (2016) showed that the recognition problem for PVGs in general is NP-hard.
- Cardinal and Hoffman (2017) concluded the problem by showing that the recognition problem for PVGs is complete in $\exists\mathbb{R}$ (exist reals).

Recognition Problem

The reduction

- We discuss the NP-hardness of the recognition problem of PVGs (Roy, 2016).

Recognition Problem

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Recognition Problem

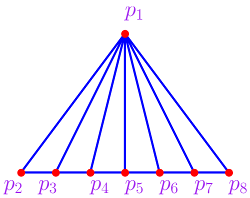
The reduction

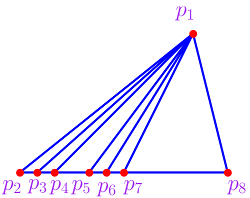
- We discuss the NP-hardness of the recognition problem of PVGs (Roy, 2016).
- For the proof, we reduce 3-SAT to the recognition problem.
- We construct two graphs, a 3-SAT graph that corresponds to a 3-SAT formula θ, and a modified slanted grid graph that has limited number of visibility embeddings.
- We combine these two graphs to form a reduction graph that is a PVG if and only if θ has a satisfying assignment.

Recognition Problem

A basic concept

A line in a visibility embedding of a *PVG* is said to be *preserved* iff it is a line containing only the same points in identical order in every visibility embedding of the *PVG*.





In the visibility embeddings of the same graph above, all lines are preserved.

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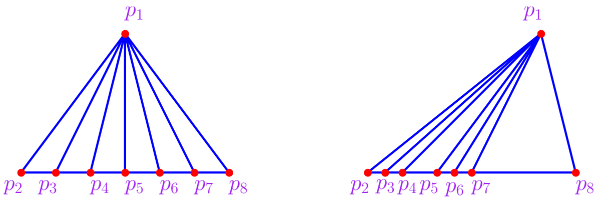
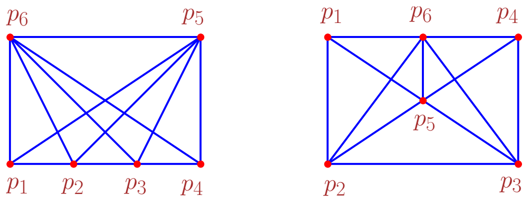


If all lines are of a visibility embedding are preserved, then the PVG is said to have a visibility embedding unique up to the preservation of lines.

Recognition Problem

A basic concept

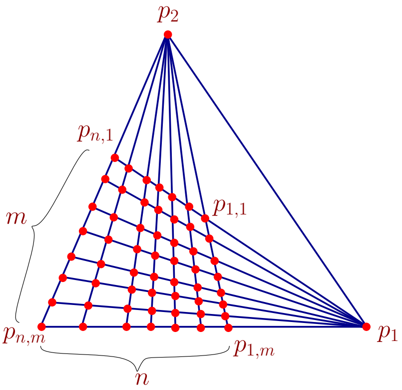
A line in a visibility embedding of a *PVG* is said to be *preserved* iff it is a line containing only the same embedding points in identical order in every visibility embedding of the *PVG*.



In the visibility embeddings of the same graph above, line $p_1p_2p_3p_4$ is not preserved.

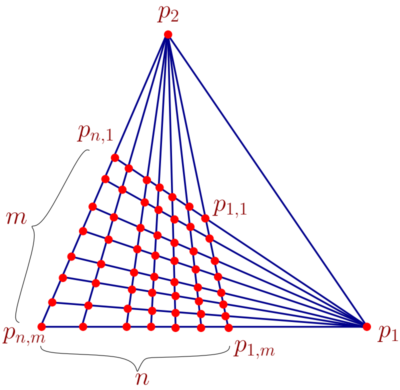
Recognition Problem

Slanted grid graph



Recognition Problem

Slanted grid graph

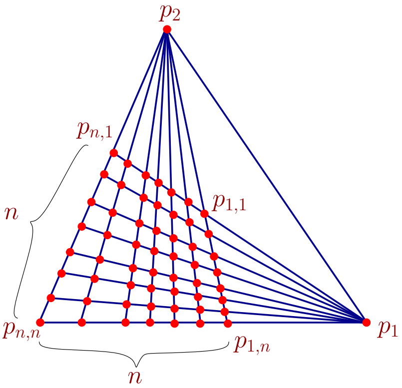


Theorem

Every slanted grid graph has a unique visibility embedding, up to the preservation of lines

Recognition Problem

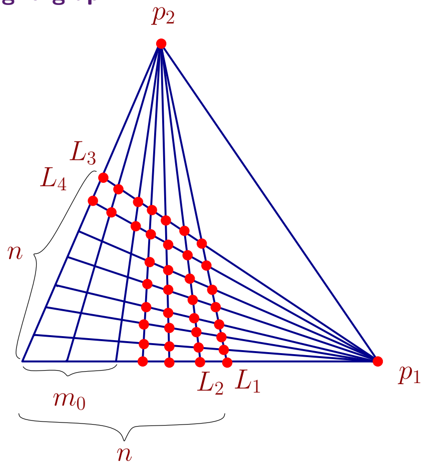
Modifying the slanted grid graph



Start with an $n \times n$ slanted grid graph.

Recognition Problem

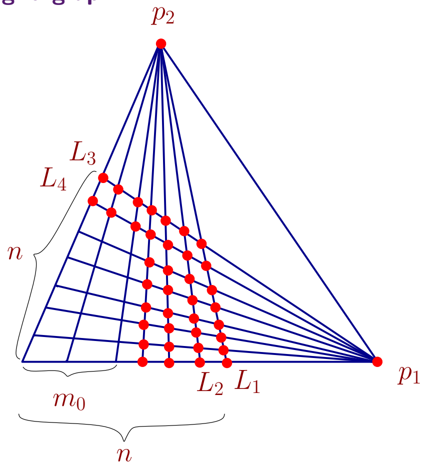
Modifying the slanted grid graph



Delete a subgrid of $m_0(n-2)$ vertices, $m_0 \leq (n-2)$, for embedding the 3-SAT graph later.

Recognition Problem

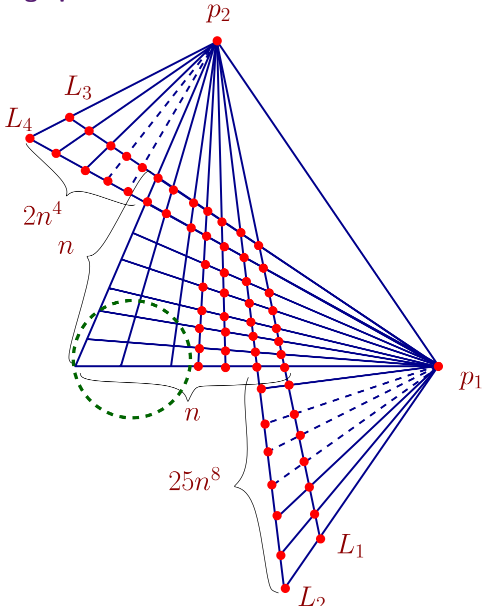
Modifying the slanted grid graph



Delete a subgrid of $m_0(n-2)$ vertices, $m_0 \leq (n-2)$, for embedding the 3-SAT graph later. Finally, to get the MSGG, add $2n^4$ vertices to the two topmost lines (L_3 and L_4) each and $25n^8$ vertices to the two rightmost lines (L_1 and L_2) each.

Recognition Problem

Modified slanted grid graph



Recognition Problem

Modified slanted grid graph

Theorem

Every modified slanted grid graph has a unique visibility embedding up to the preservation of lines.

The proof of this lemma consists of a strengthening of the following lemma.

Lemma

Let G be a PVG with visibility embedding ξ. Let L be a line in ξ such that (i) there are l points on L, and (ii) k points not on L. If $l \geq (k+3)^2$ then L is preserved in every visibility embedding of G

Modified slanted grid graph

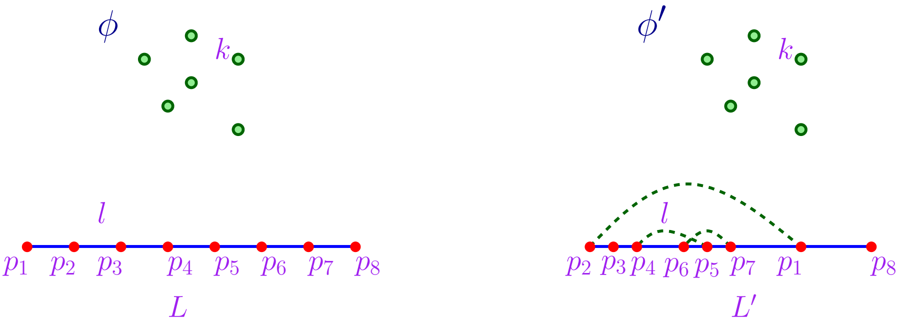
Proof sketch

Let the initial embedding of G containing L be ϕ. Consider another visibility embedding ϕ' where L is not preserved. There can be the following cases for ϕ':

1. All points of L are collinear in ϕ'.
   - a Their order is not preserved.
   - b Their order is the same, but som other points are also collinear with them.
2. Not all points of L are collinear in ϕ'.
   - a At least some $k+3$ points of L are collinear in ϕ'.
   - b At most $k+2$ points of L are collinear in ϕ'.

Modified slanted grid graph

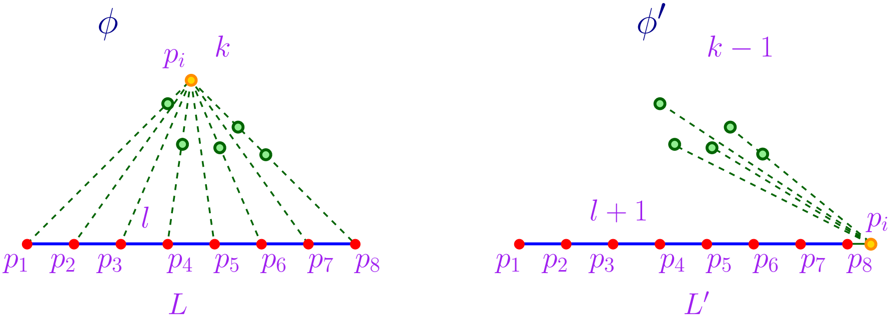
Proof sketch



Case 1 a: All points of L are collinear in ϕ', but their order is not preserved. This is not possible because some visibilities disappear while some new visibilities appear.

Modified slanted grid graph

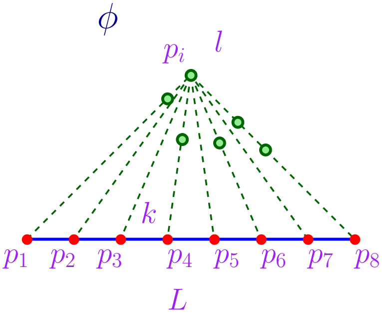
Proof sketch

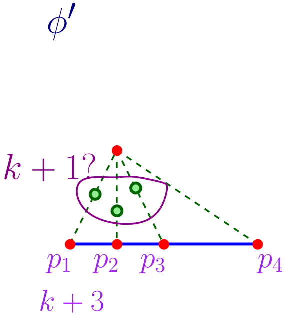


Case 1 b: All points of L are collinear in ϕ' in the same order, but some other points are also collinear with them. This is not possible because any point outside of L must see at least l points, while any point on L can see only at most $k+2$ points.

Modified slanted grid graph

Proof sketch

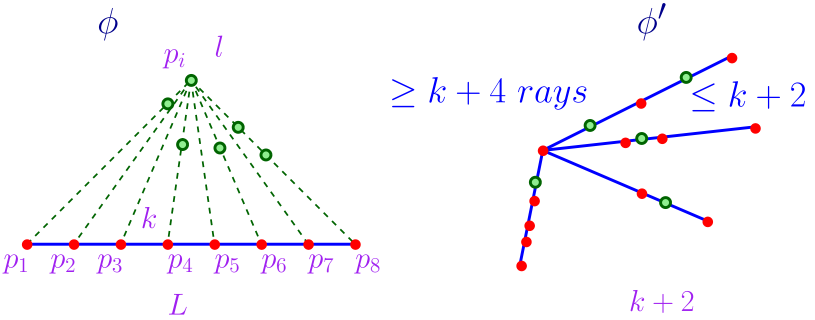




Case 2 a: Not all points of L are collinear in ϕ', and at least some $k+3$ points of L are collinear in ϕ' (say, on line L'). The point of L closest to this L' sees at least $k+3$ points. Since this point can see at most 2 points of L, it must see $k+1$ points that are not in L.

Modified slanted grid graph

Proof sketch



Case 2 b: Not all points of L are collinear in ϕ', and at most $k+2$ points of L are collinear in ϕ'. Then a point of L must have at least $k+4$ rays emanating from it containing all the other points of L. So the point must see at least $k+2$ points not on L.

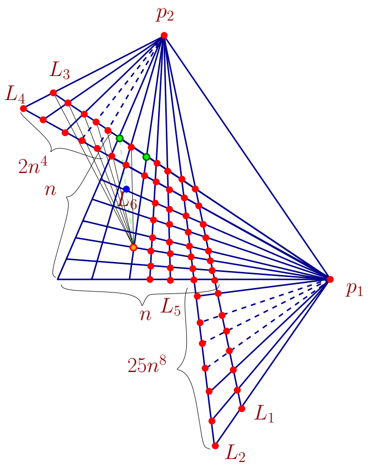
Modified slanted grid graph

- A modified form of this lemma is applied twice on an MSGG to give the proof of the theorem. Hence the MSGG has $O(n^8)$ vertices, as the original grid has $O(n^2)$ vertices.

Modified slanted grid graph

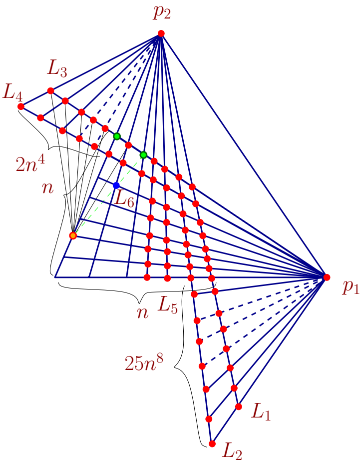
- A modified form of this lemma is applied twice on an MSGG to give the proof of the theorem. Hence the MSGG has $O(n^8)$ vertices, as the original grid has $O(n^2)$ vertices.
- The deleted subgrid of the MSGG can be replaced with another gadget, such that the resultant graph can have only a limited number of visibility embeddings.

Modified slanted grid graph



The position of a point of the gadget can be controlled by its visibility relationship with the MSGG.

Modified slanted grid graph



The position of a point of the gadget can be controlled by its visibility relationship with the MSGG.

Recognition Problem

3-SAT graph

1. We construct a gadget called the *3-SAT graph* to embed in the deleted subgrid of an MSGG.

2. For any given 3-SAT formula θ, a 3-SAT graph of polynomial size can be contructed in polynomial time.

3. The 3-SAT graph can be strategically embedded in a large enough MSGG, which is again of polynomial size with respect to the size of the 3-SAT formula.

Recognition Problem
3-SAT graph

- Once combined with the MSGG, each point of the 3-SAT graph can be embedded only in a definite horizontal line of the grid, while there might be a choice for the vertical line.
- After the 3-SAT graph is combined with the MSGG, it is divided into vertical strips called variable regions and clause regions, corresponding to the variables and clauses of θ.
- The red points represent the assignment of 0 or 1 to a variable in θ. A red point is embedded in the left of a variable region if the variable is assigned 1, otherwise it is embedded in the middle of a variable region if the variable is assigned 0.

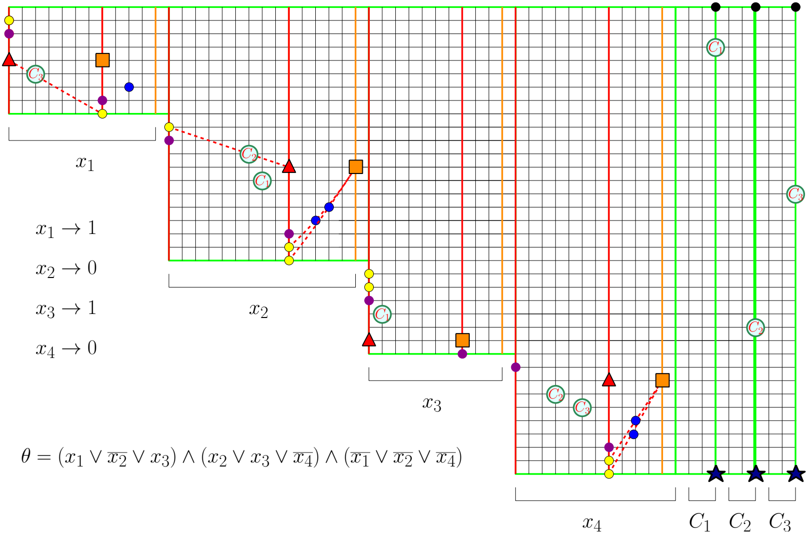
Recognition Problem

3-SAT graph

- According to their placement, each red point is to be blocked from some yellow points that represent the occurrance of a variable in a clause.
- These blockings are possible only by green points, that can be embedded only on two vertical lines, one in a variable region and the other in a clause region.
- Each clause region has a blue point that needs to be blocked from a black point vertically above it. This blocker must also be a green point. A visibility embedding is possible only when each blue point has at least one green point as a blocker. If an assignment is not satisfying then some blue point does not have a green point as a blocker.

Recognition Problem

3-SAT graph



$x_1 \to 1$

$x_2 \to 0$

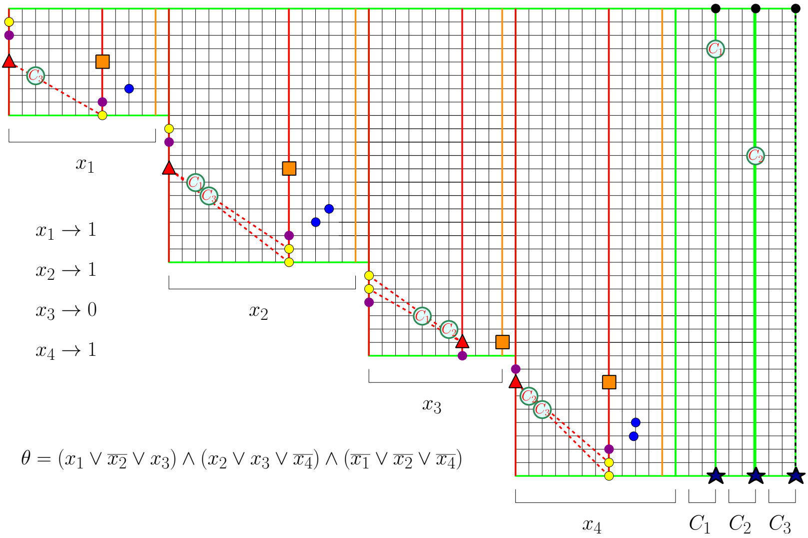
$x_3 \to 1$

$x_4 \to 0$

$$\theta = (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4})$$

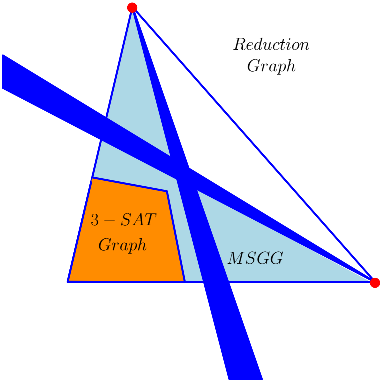
Recognition Problem

3-SAT graph



Recognition Problem

Reduction graph



The reduction graph has a visibility embedding if and only if the corresponding 3-SAT formula has a satisfying assignment.

Theorem

The point visibility graph recognition problem is NP-hard.

Optimization problems

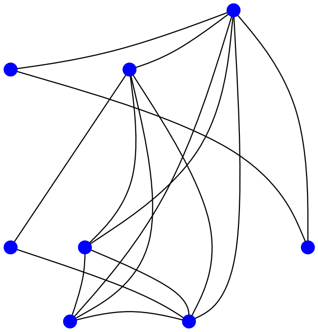
We consider graph optimization problems defined on the PVG of a given set of points in the plane.

Theorem

The problems of Vertex Cover, Independent Set and Maximum Clique remain NP- hard on point visibility graphs.

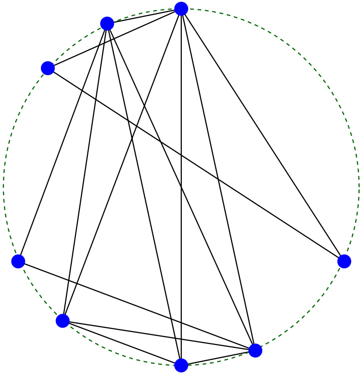
Optimization problems

Given a graph $G(V, E)$, we transform it to a PVG $G'(V', E')$.



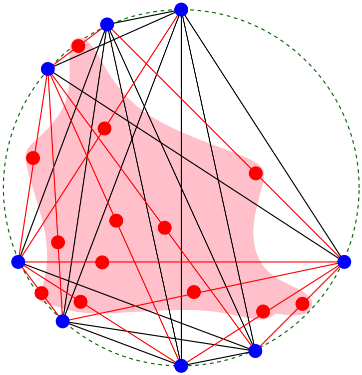
Optimization problems

We embed the vertices of G on a circle.



Optimization problems

We add points so that the new points see every point of G'.



Optimization problems

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Optimization problems

- Let $k = |V'| - |V|$.
- $|MinVC(G)| = |MinVC(G')| + k$
- $|MaxIndSet(G)| = |MaxIndSet(G')|$
- $|MaxClique(G)| = |MaxClique(G')| + k$
- Vertex Cover, Independent Set and Max-Clique are NP-hard problems on PVGs (Ghosh and Roy, 2015).

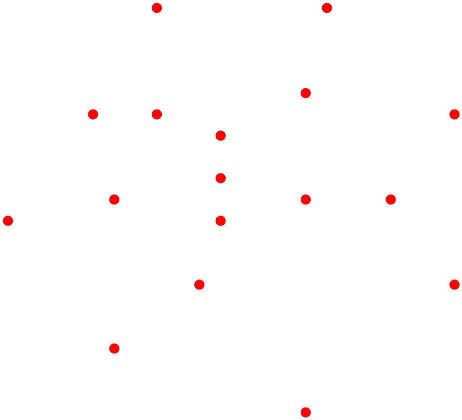
Optimization problems

- Colouring a graph is NP-hard. In fact, 3-colouring a graph is NP-hard.
- Kara et al. (2004) characterized the PVGs that can be 3-coloured, and hence gave a polynomial algorithm for 3-colouring PVGs.
- Diwan and Roy (2017) showed that 5-colouring PVGs is NP-hard.

Optimization problems

- Hamiltonian cycle is NP-hard on general graphs.

Optimization problems



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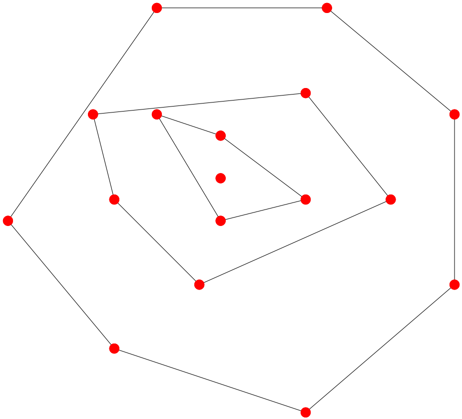

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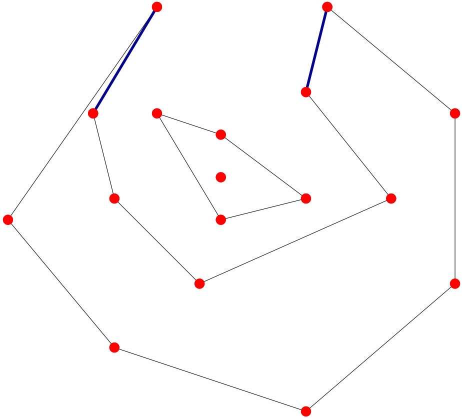

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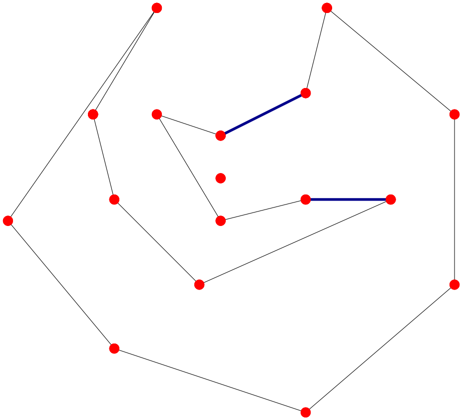

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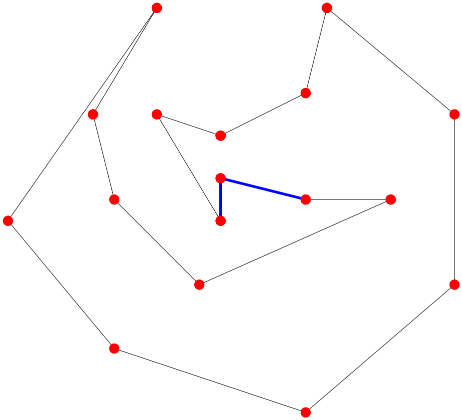


Optimization problems

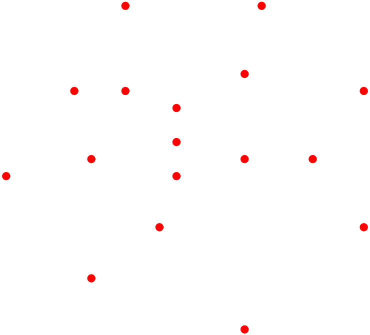
- If G is a PVG but not a path, then G has a Hamiltonian cycle.

- Given G and a visibility embedding of G, a Hamiltonian cycle in G can be constructed in linear time (Ghosh and Roy, 2015).

  The whole graph including edges is given as input, and so the gift-wrapping algorithm takes only linear time, as we first find one convex hull vertex and then go through its edge list to find the next convex hull vertex.

Disjoint triangle partition

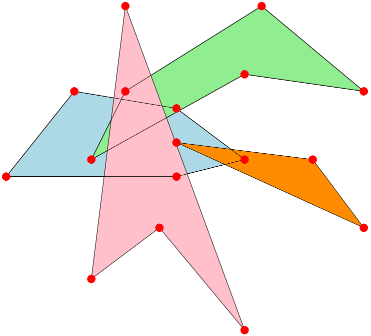
Basic concepts



$P = \{p_1, p_2, \ldots, p_n\}$ is a given set of points in the plane.

Disjoint triangle partition

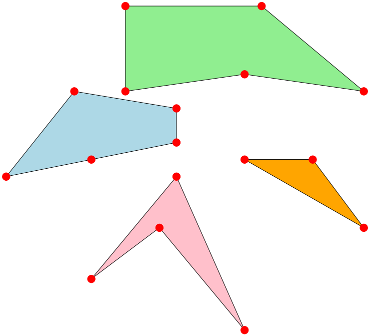
Basic concepts



A partition of P into subsets $S_1, S_2, \ldots, S_j$ is a *cycle partition* of P, when the points of each S_i can be joined by straight line segments to form a simple polygon i.e. no line contains all points of S_i.

Disjoint triangle partition

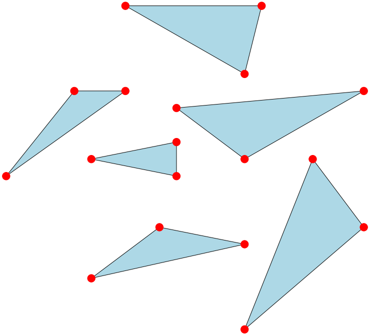
Basic concepts



A cycle partition of a point set is *disjoint* when no two of the polygons enclosed by the cycles intersect with respect to vertices, edges or area.

Disjoint triangle partition

Basic concepts



A *disjoint triangle partition* of P is a disjoint cycle partition of P where all the cycles are triangles.

Disjoint triangle partition

Basic concepts

- We characterize planar point sets that admit a disjoint triangle partition and provide a polynomial time algorithm to construct such a partition, if it exists.

Disjoint triangle partition

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- We characterize planar point sets that admit a disjoint triangle partition and provide a polynomial time algorithm to construct such a partition, if it exists.
- Given a set S of cycles, we characterize planar point sets that admit a disjoint partition into cycles of S and provide a polynomial time algorithm to construct such a partition, if it exists.

Disjoint triangle partition

We say that two points p_i and p_j of P are *visible* to each other if the line segment $p_i p_j$ does not contain any other point of P.

A subset I of P such that no two points of I are visible from each other is called an *independent set* of P.

Disjoint triangle partition

Theorem

A set P of $3n$ points in the plane admits a disjoint triangle partition iff P does not contain an independent set of size $n+1$.

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The proof largely depends on identifying the point sets that contain an independent set of size $n+1$.

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The proof largely depends on identifying the point sets that contain an independent set of size $n+1$.

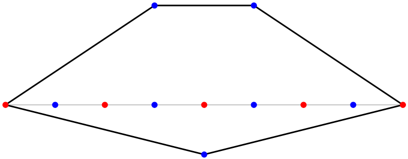
Lemma

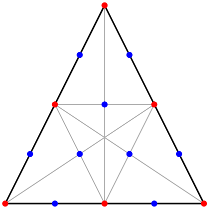
Let P be a set of $3n$ points that contains an independent set I of size $n+1$. Then one of the following must hold:

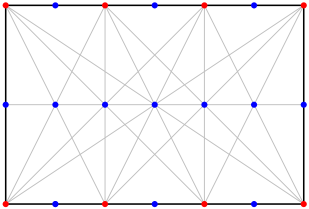
1. *The points in I are collinear.*
2. *The points in I occur on the boundary of $CH(P)$ and $CH(I) = CH(P)$. $CH(P)$ has at most 4 vertices and the boundary of $CH(P)$ contains exactly $2n+2$ points of P , with every alternate point in I. Further, every subset of 5 points in I must contain 3 collinear points.*

Disjoint triangle partition

Some forbidden point sets.







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Proof sketch: Suppose P admits a disjoint triangle partition and contains an independent set of size $n+1$. Then some two points in the independent set must be in the same triangle in the triangle partition. Since the triangles are disjoint, these two points must be visible to each other, contradicting the fact that they are in an independent set.

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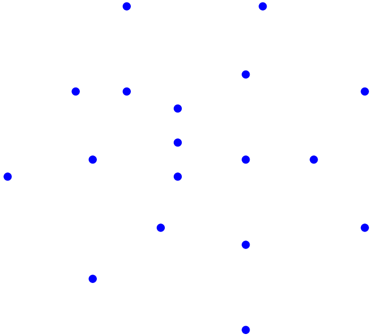
Suppose P does not contain an independent set of size $n+1$. We show that P has a disjoint triangle partition. The proof is by induction on n. For $n=1$, this is trivial. Suppose $n \geq 2$.

Disjoint triangle partition

Let p_i be any vertex of $CH(P)$, p_j the point in P that follows p_i on the boundary of $CH(P)$ in clockwise order, and p_k the point that precedes p_j on the boundary of $CH(P \setminus \{p_i\})$.



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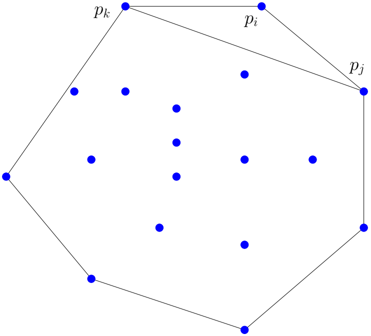
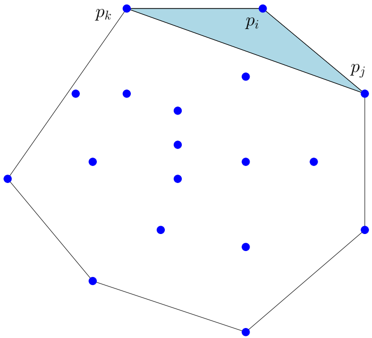


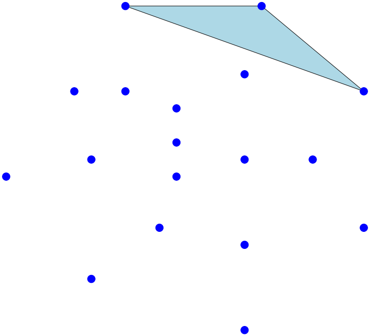
Disjoint triangle partition

Call this triangle $p_i p_j p_k$ as $\Delta(p_i)$



Disjoint triangle partition

Let $P' = P \setminus \{p_i, p_j, p_k\}$. If If P' does not contain an independent set of size n, then by induction, we are done.



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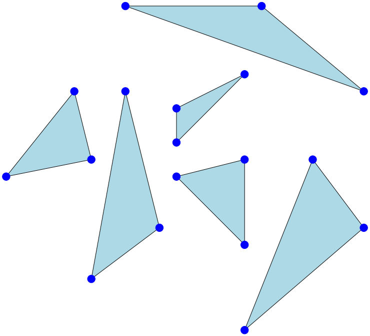
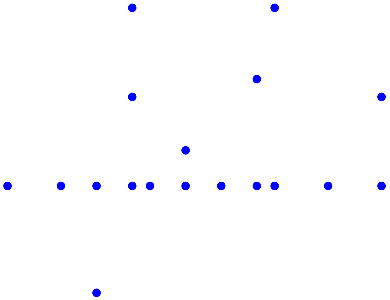


Disjoint triangle partition

Else suppose P' contains $2n-1$ collinear points on some line L.



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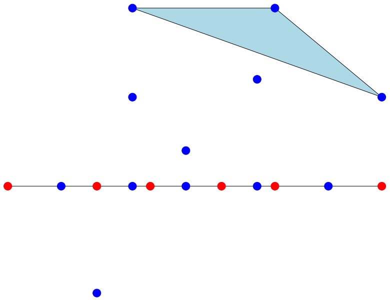
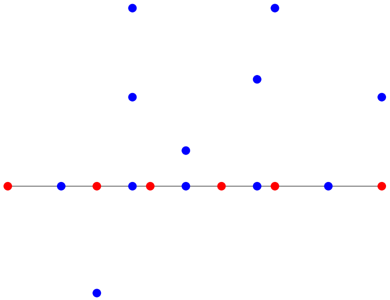


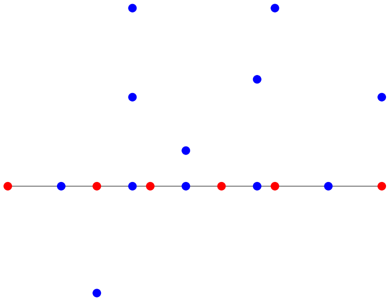
Disjoint triangle partition

Let these be $p_1, p_2, \ldots, p_{2n}$ in left to right order along L.



Disjoint triangle partition

For $i = 1$ to n, we choose the triangle p_{2i-1}, p_{2i}, q_i, where q_i is a point in P not in L, that has not been included in any earlier triangle, such that the angle p_{2i-1}, p_{2i}, q_i is as small as possible, and subject to this condition, q_i is as close to p_{2i} as possible.



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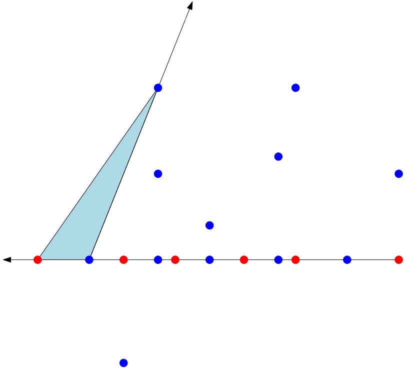

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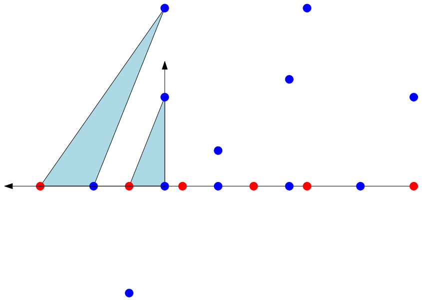

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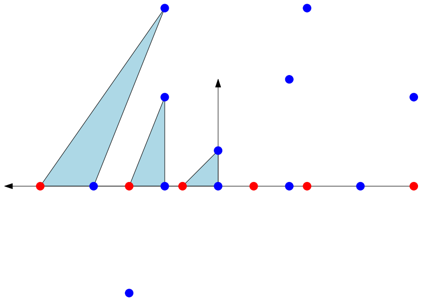

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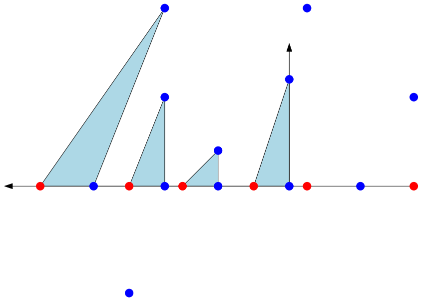

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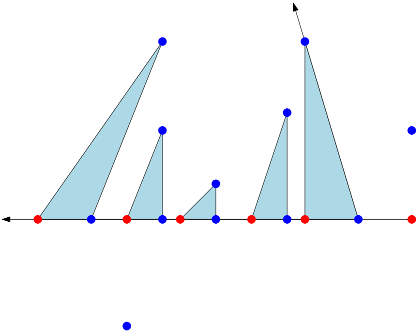
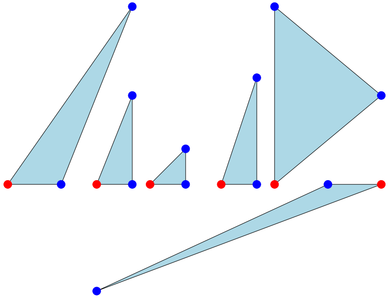


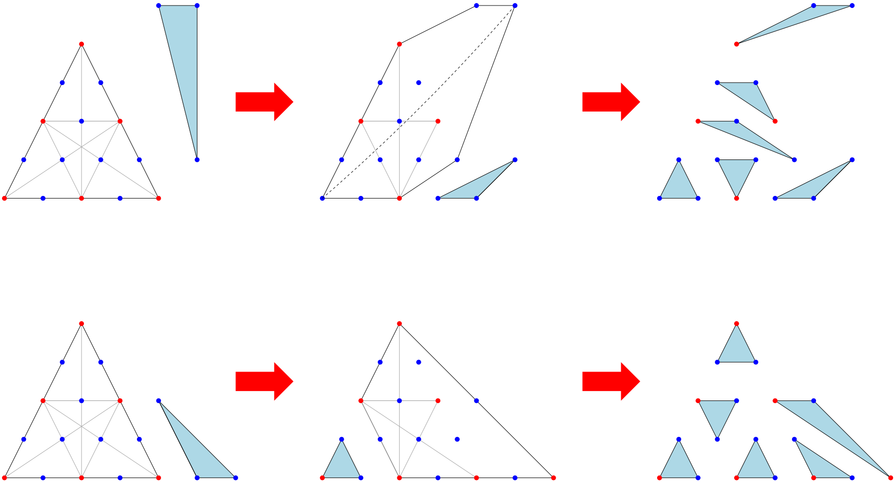
Disjoint triangle partition

If the remaining two points are on different sides of L then we swap a point with the last constructed triangle.



Disjoint triangle partition

If $CH(P')$ satisfies the second condition, then we can conveniently choose an alternative triangle to obtain another subset of P that does not satisfy the condition.



Generalized cycle partition

Theorem

Let $C_1, C_2, \ldots, C_k$ be a collection of cycles of lengths $L_1, L_2, \ldots, L_k$ such that $L_k \geq 4$. A set P of $L = \sum_{i=1}^{k} L_i$ points admits a disjoint cycle partition into cycles of lengths $L_1, L_2, \ldots, L_k$ iff it does not contain $L - k + 1$ collinear points.

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Lemma

If P has $\sum_{i=1}^{k} L_i$ points, not all collinear, then it is possible to separate out C_i from P so that C_i and $CH(P \setminus C_i)$ are disjoint.

Generalized cycle partition

The theorem is proved using the Lemma, subtracting individual cycles from P and then partitioning the remaining points at once when conveniently large collinearities occur.

Generalized cycle partition

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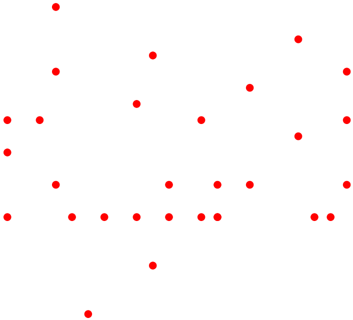
Proof Sketch: If P contains $L - k + 1$ collinear points on some line, then there are at most $k - 1$ points of P not in the line. Thus in any partition of P into k parts, some part must contain all points in the line. Thus P cannot have a cycle partition into k cycles of lengths $L_1, \ldots, L_k$.

General cycle partitions

Now we proceed by induction. Let P be a set of points.

$S = \{4, 8, \underset{p}{6}, 4, 3, 4\}$



General cycle partitions

Let p be any vertex of $CH(P)$, p_0 the point of P that precedes p on the boundary of $CH(P)$, and q the point that follows p.

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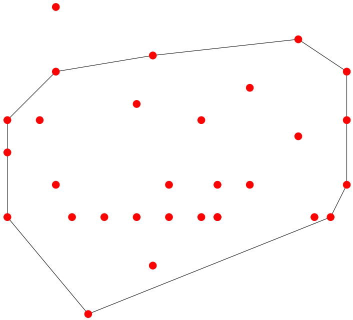

General cycle partitions

Then p_0 and q are vertices of $CH(P \setminus \{p\})$ and let $p_0, p_1, p_2, \ldots, p_k = q$ be the points of P that occur between p_0 and q on the boundary of $CH(P \setminus \{p\}$.

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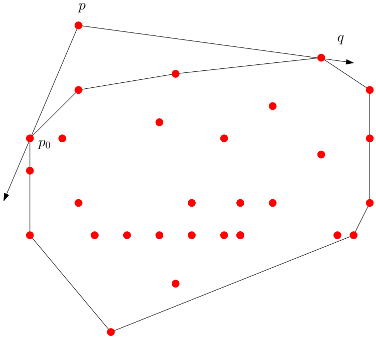
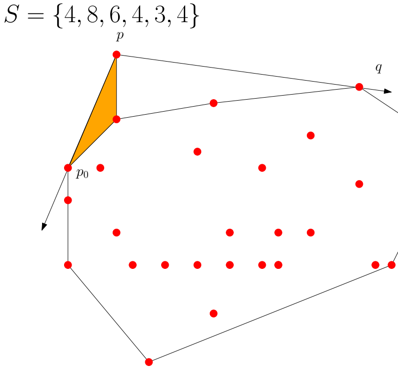


General cycle partitions

If $k \geq L_i - 2$, we choose C_i to be the cycle $p, p_0, p_1, p_2, \ldots, p_{L_i-2}$.



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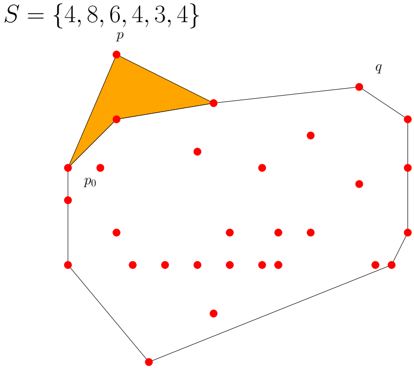

General cycle partitions

If $k < L_i - 2$, then $L_i > 3$. We delete the points $p, p_1, \ldots, p_{k-1}$, and in the remaining set of P' of points, find a cycle C_i' of length $L_i - k$, using the same procedure, starting with the vertex q.

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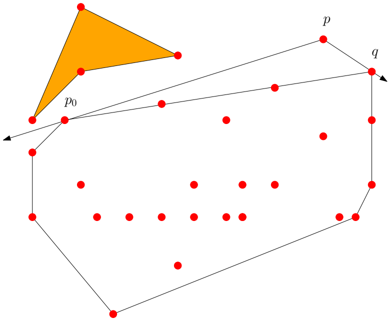

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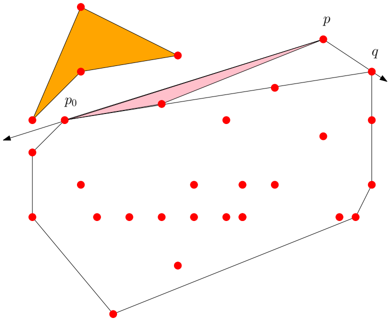

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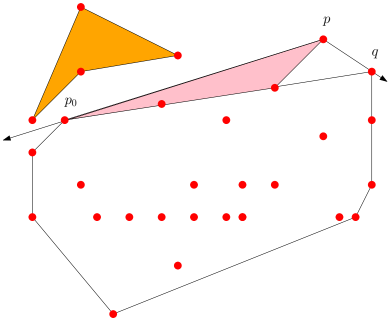

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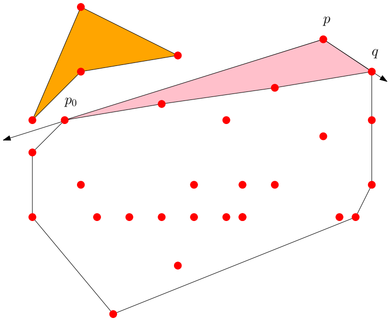

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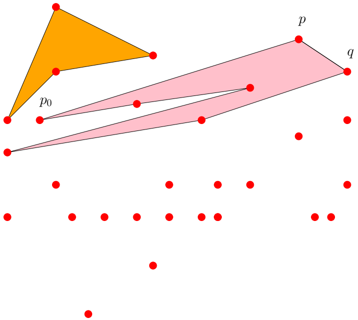

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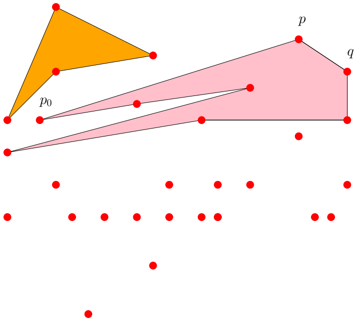

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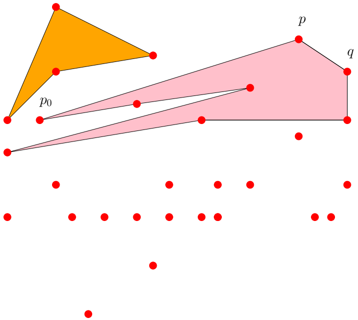

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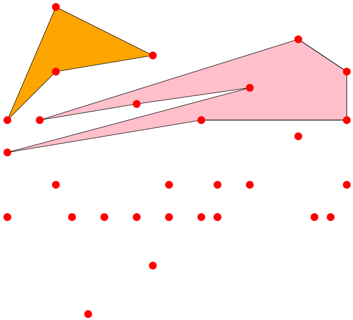

General cycle partitions

We continue this process. Suppose at some stage, among the remaining points, the condition of the theorem is violated, and there are $L - k + 1$ collinear points.

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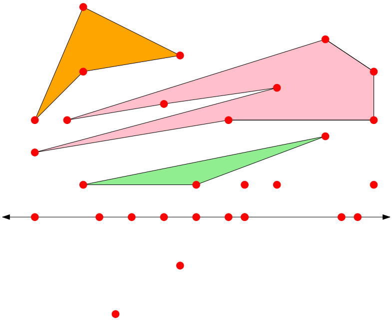

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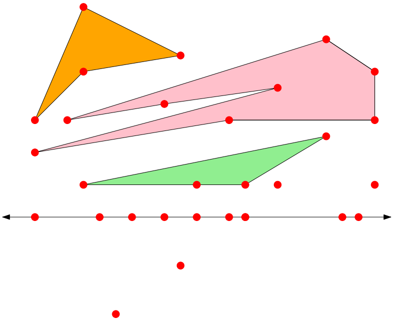

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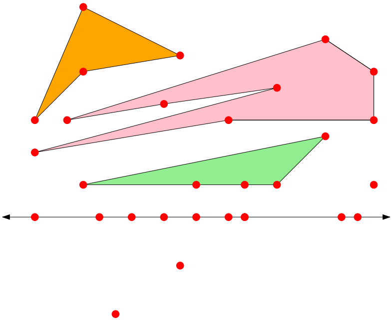

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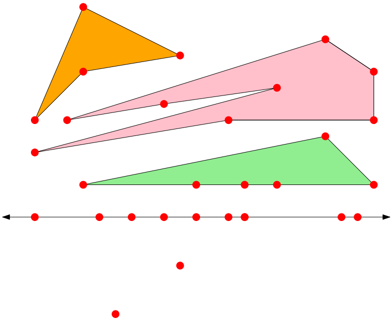
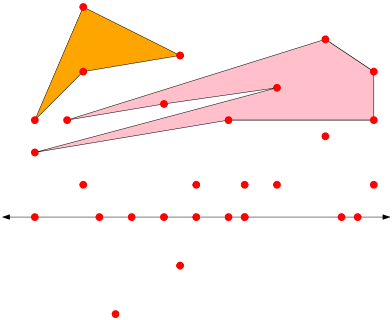


General cycle partitions

Then we erase the last constructed triangle.

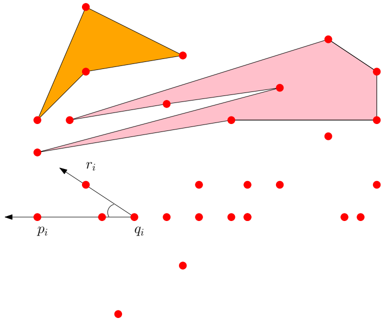
$S = \{4, 8, 6, 4, 3, 4\}$



General cycle partitions

We continue in a method similar to the first case of the triangle partition method, were a large number of points are collinear.

$S = \{4, 8, 6, 4, 3, 4\}$



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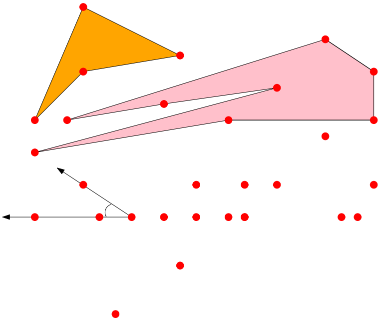

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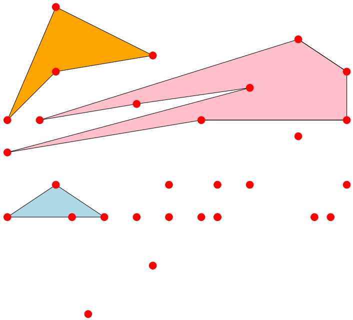

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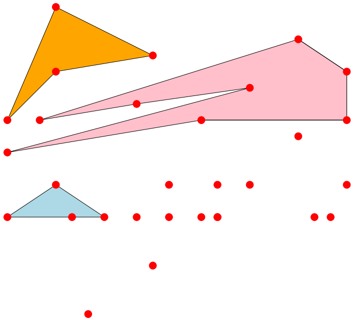

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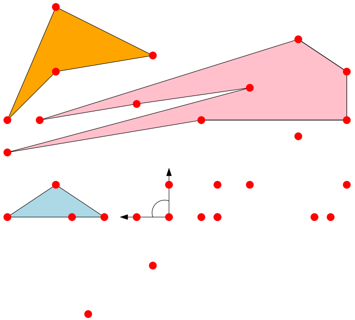

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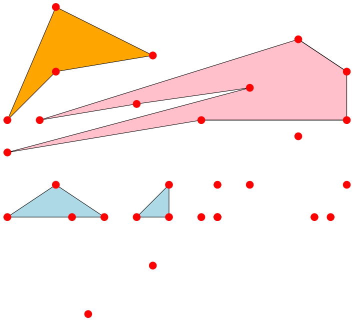

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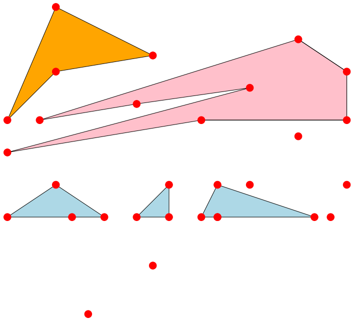
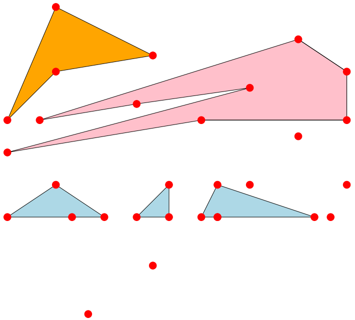


General cycle partitions

Suppose we are left with only one point on the line.

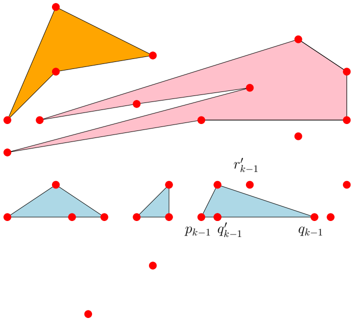
$S = \{4, 8, 6, 4, 3, 4\}$



General cycle partitions

We swap a point with the last constructed cycle, which is why it must have been of length at least 4.

$S = \{4, 8, 6, 4, 3, 4\}$



General cycle partitions

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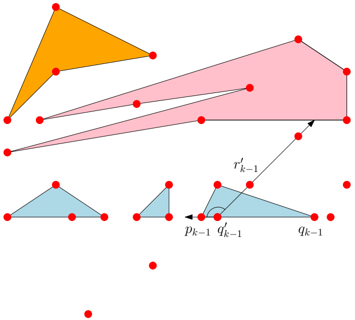
$S = \{4, 8, 6, 4, 3, 4\}$



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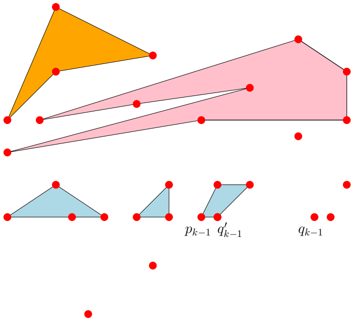
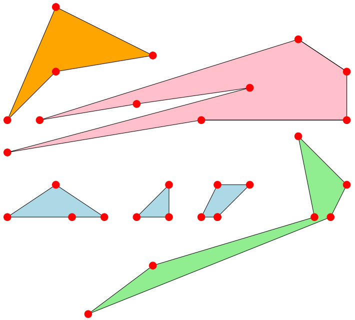


General cycle partitions

This completes the disjoint cycle partition.

$S = \{4, 8, 6, 4, 3, 4\}$



Open problems

- The complexity of Dominating Set on PVGs is unknown.
- The disjoint cycle partition problem where each cycle is a convex polygon, is yet to be solved.

Thank You!