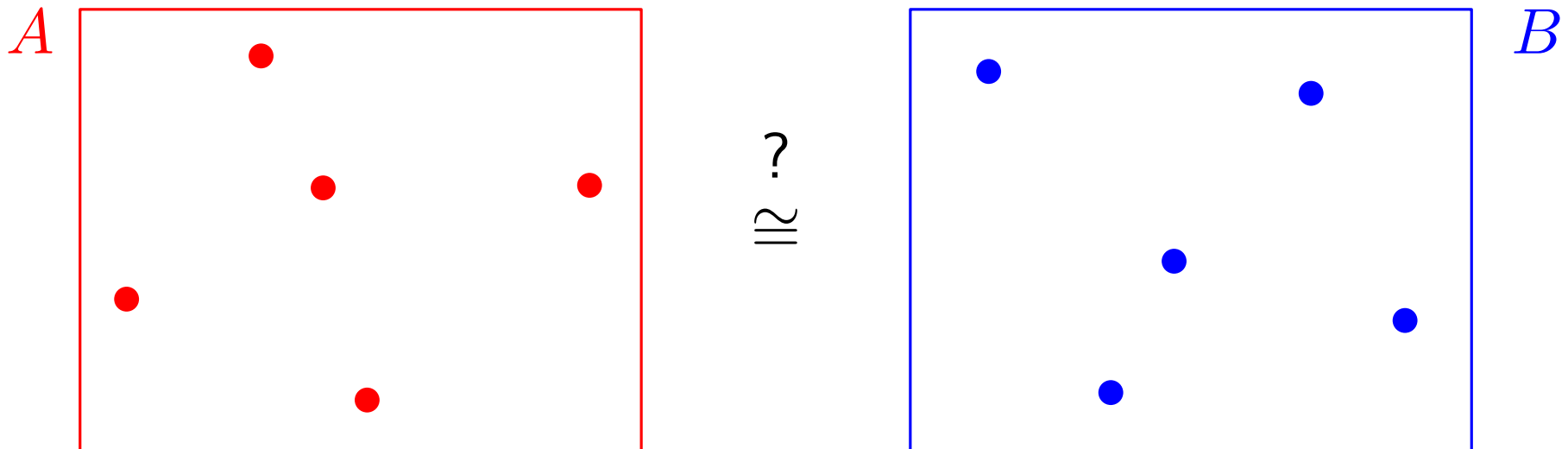


# Congruence Testing in 4 Dimensions

Günter Rote

joint work with Heuna Kim

Freie Universität Berlin

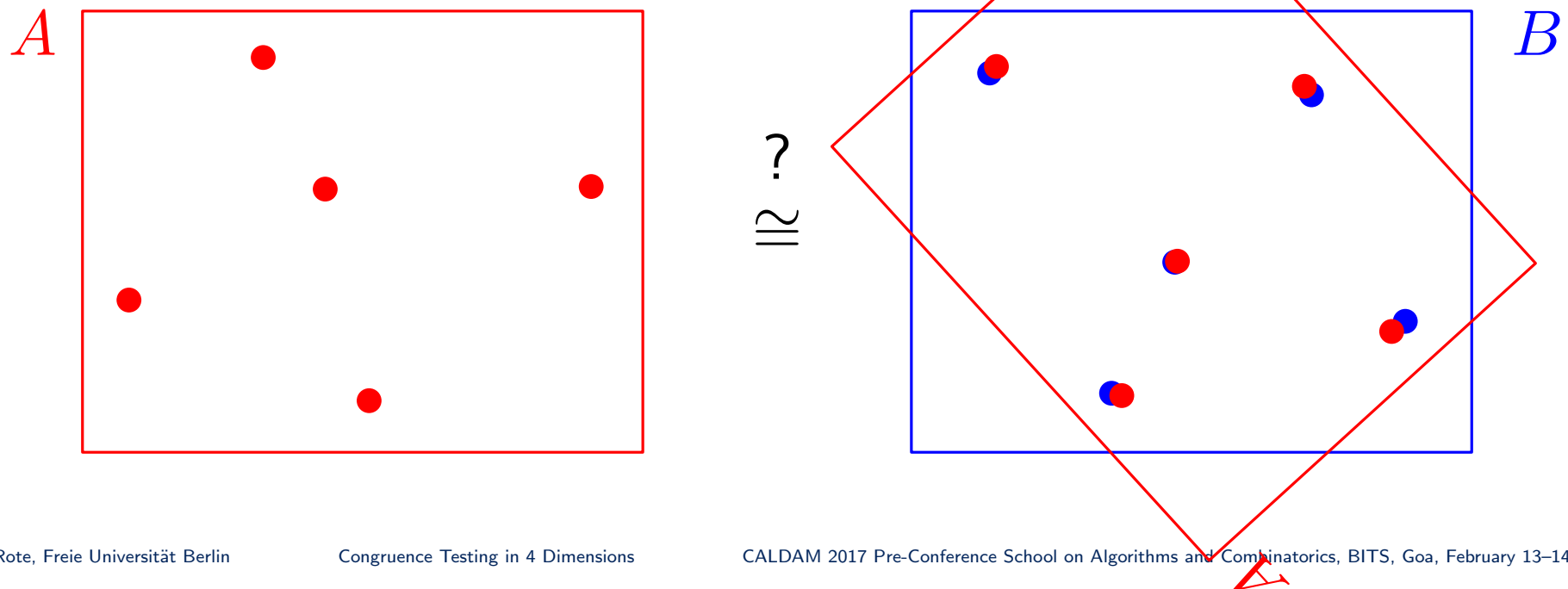


# Congruence Testing in 4 Dimensions

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- 1 dimension
- 2 dimensions
- 3 dimensions
- 4 dimensions
- $d$  dimensions

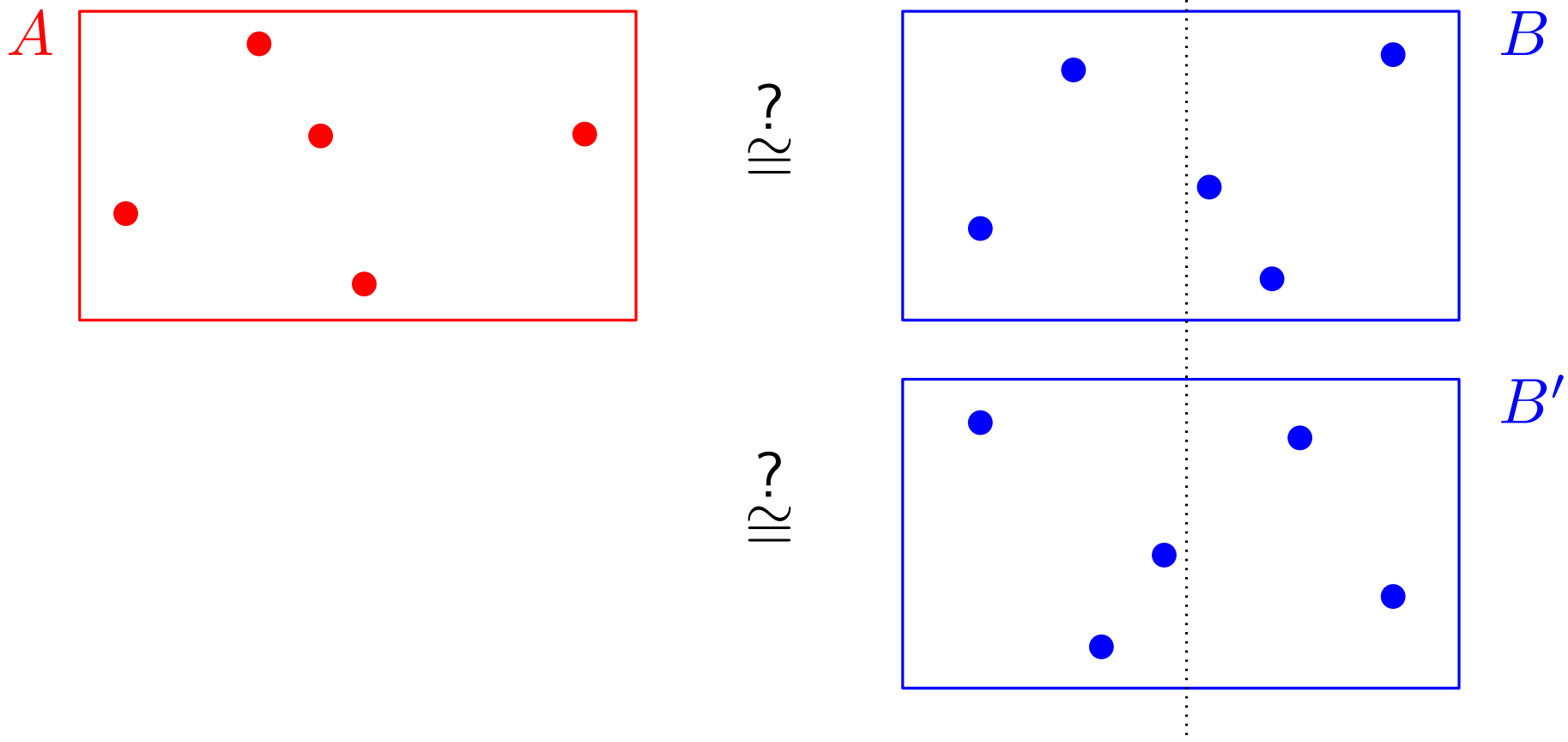
- 1 dimension
  - 2 dimensions
  - 3 dimensions
  - 4 dimensions
  - $d$  dimensions
- $\left. \begin{array}{l} \bullet 1 \text{ dimension} \\ \bullet 2 \text{ dimensions} \\ \bullet 3 \text{ dimensions} \end{array} \right\} O(n \log n) \text{ time}$
- $\bullet 4 \text{ dimensions} \quad \left. \begin{array}{l} \bullet 3 \text{ dimensions} \\ \bullet 4 \text{ dimensions} \end{array} \right\} \text{NEW, joint work with Heuna Kim}$
- $\bullet d \text{ dimensions} \quad O(n^{\lceil d/3 \rceil} \log n) \text{ time [Brass and Knauer 2002]}$
- $O(n^{\lfloor d/2 \rfloor / 2} \log n) \text{ time Monte Carlo [Akutsu 1998/Matoušek]}$

- 1 dimension
  - 2 dimensions
  - 3 dimensions
  - 4 dimensions
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- $O(n^{\lceil d/3 \rceil} \log n)$  time [Brass and Knauer 2002]
- $O(n^{\lfloor d/2 \rfloor / 2} \log n)$  time Monte Carlo [Akutsu 1998/Matoušek]
- Problem statement and variations
  - Dimension reduction as in [Alt, Mehlhorn, Wagener, Welzl]
  - Atkinson's reduction (pruning/condensation)
  - (Planar) graph isomorphism
  - Hopf fibrations
  - Plücker coordinates
  - Coxeter groups

# Rotation or Rotation+Reflection?

We only need to consider *proper* congruence (orientation-preserving congruence, of determinant  $+1$ ).

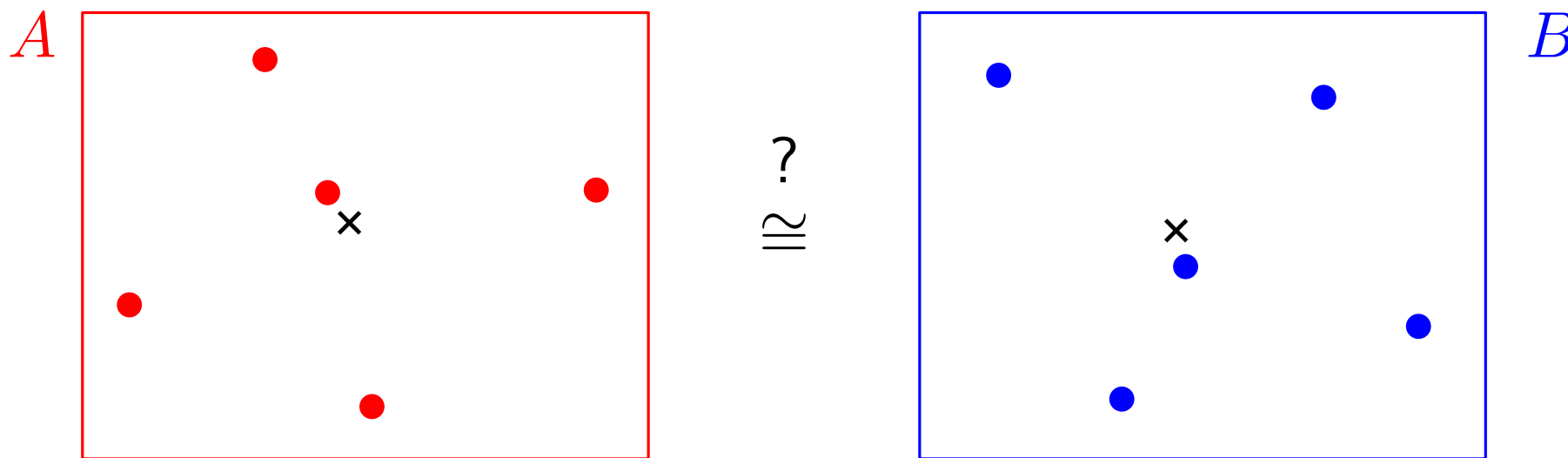
If mirror-congruence is also desired, repeat the test twice, for  $B$  and its mirror image  $B'$ .



# Congruence = Rotation + Translation

Translation is easy to determine:

The centroid of  $A$  must coincide with the centroid of  $B$ .



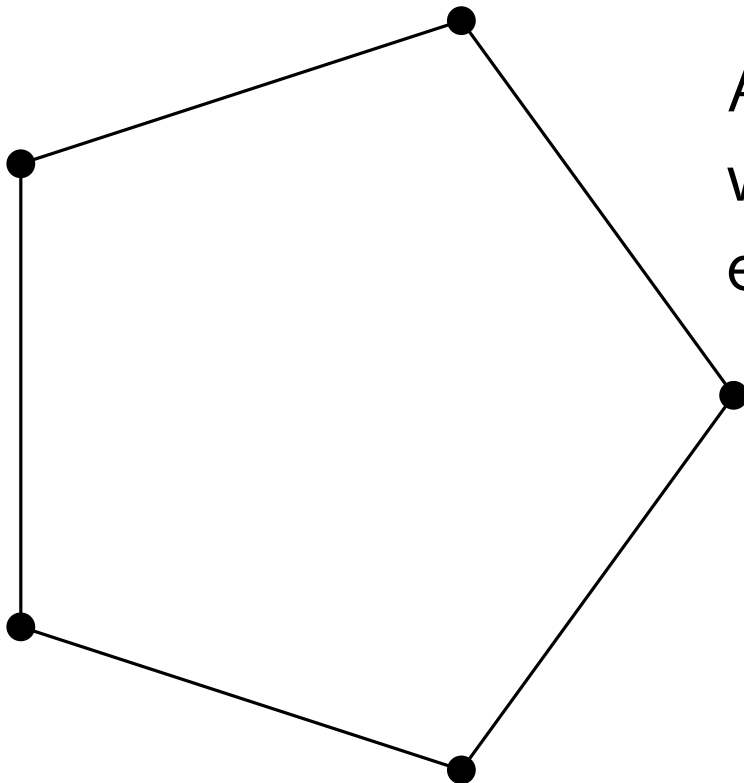
→ from now on: All point sets are centered at the origin 0:

$$\sum_{a \in A} a = \sum_{b \in B} b = 0$$

We need to find a rotation around the origin (orthogonal matrix  $T$  with determinant  $+1$ ) which maps  $A$  to  $B$ :  $TA = B$

The proper setting for this (mathematical) problem requires real numbers as inputs and exact arithmetic.

→ the *Real RAM* model (RAM = random access machine):  
One elementary operation with real numbers ( $+$ ,  $\div$ ,  $\sqrt{\phantom{x}}$ ,  $\sin$ ) is counted as one step.



A regular 5-gon, 7-gon, 8-gon, ...  
with rational coordinates does not  
exist in any dimension.

Congruence testing is the basic problem for many pattern matching tasks

- computer vision
- star matching
- brain matching
- . . .

The proper setting for this applied problem requires tolerances, partial matchings, and other extensions.

$A, B \subset \mathbb{R}^d$ ,  $|A| = |B| = n$ .

We consider the problem for fixed dimension  $d$ .

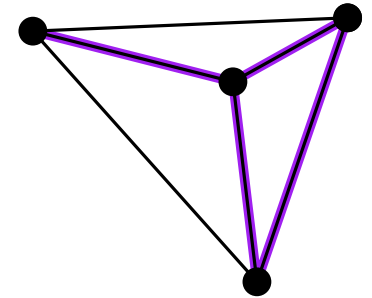
When  $d$  is unrestricted, the problem is equivalent to **graph isomorphism**:

$G = (V, E)$ ,  $V = \{1, 2, \dots, n\}$

$\mapsto A = \underbrace{\{e_1, \dots, e_n\}}_{\text{regular simplex}} \cup \left\{ \frac{e_i + e_j}{2} \mid ij \in E \right\} \subset \mathbb{R}^n$

regular simplex

$e_i = (0, \dots, 0, 1, 0, \dots, 0)$



CONJECTURE:

Congruence can be tested in  $O(n \log n)$  time for every fixed dimension  $d$ .

Current best bound:  $O(n^{\lceil d/3 \rceil} \log n)$  time

$A, B \subset \mathbb{R}^d$ ,  $|A| = |B| = n$ .

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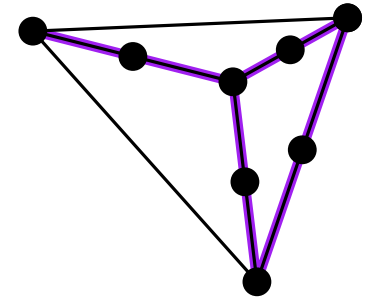
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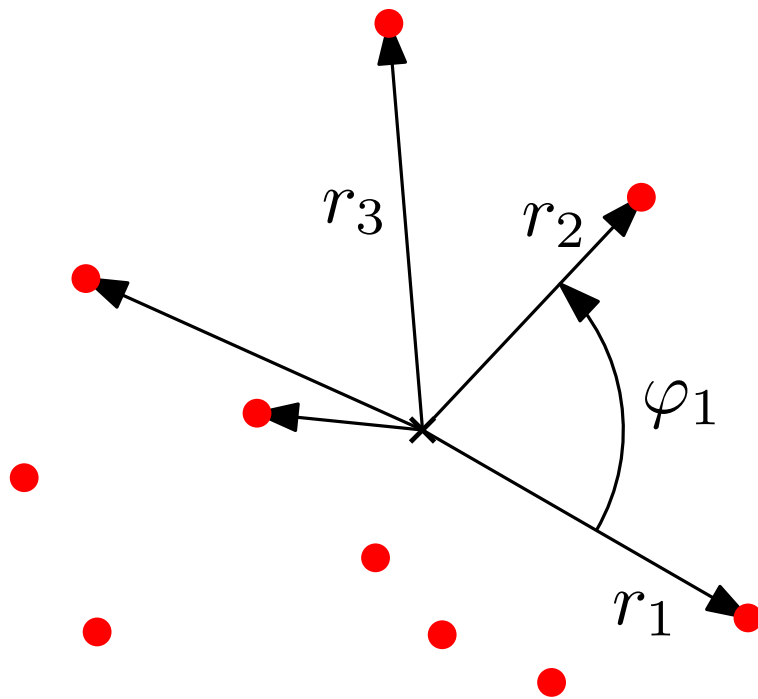
Current best bound:  $O(n^{\lceil d/3 \rceil} \log n)$  time

Can be done by string matching.

[ Manacher 1976 ]

Sort points around the origin.

Encode alternating sequence of distances  $r_i$  and angles  $\varphi_i$ .



$$(r_1, \varphi_1, r_2, \varphi_2, \dots, r_n, \varphi_n)$$

Check whether the corresponding sequence of  $B$  is a cyclic shift.

$\rightarrow O(n \log n) + O(n)$  time.

Can be done by string matching.

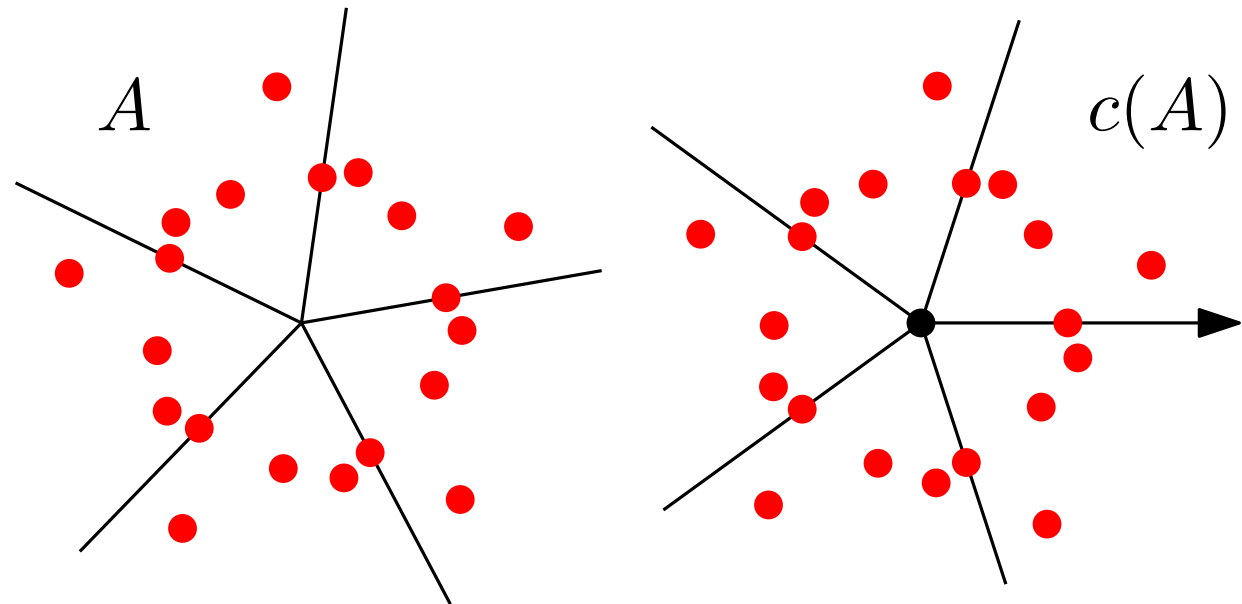
[ Manacher 1976 ]

Sort points around the origin.

Encode alternating sequence of distances  $r_i$  and angles  $\varphi_i$ .

Even more  
can be done:

## CANONICAL DIRECTIONS



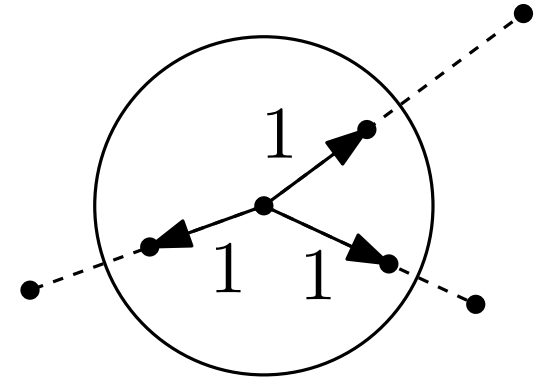
The *canonical set*  $c(A)$ : [Akutsu 1992]

$$A \cong B \iff c(A) = c(B)$$

→ searching in a database

[ Sugihara 1984; Alt, Mehlhorn, Wagener, Welzl 1988 ]

Project points to the unit sphere,  
and keep distances as *labels*.

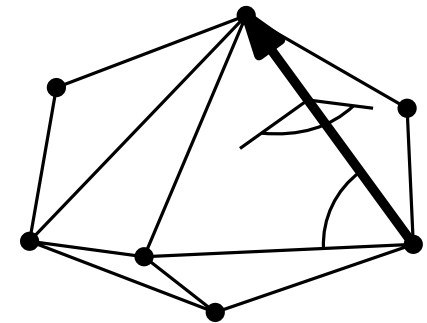


Compute the convex hulls  $P(A)$  and  $P(B)$ , in  $O(n \log n)$  time.

Check isomorphism between the corresponding LABELED planar graphs.

Vertex labels: from the radial projection

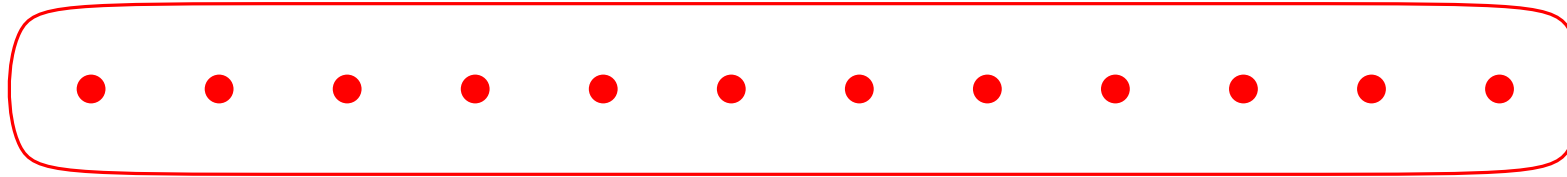
Edge labels: dihedral angles and face angles.



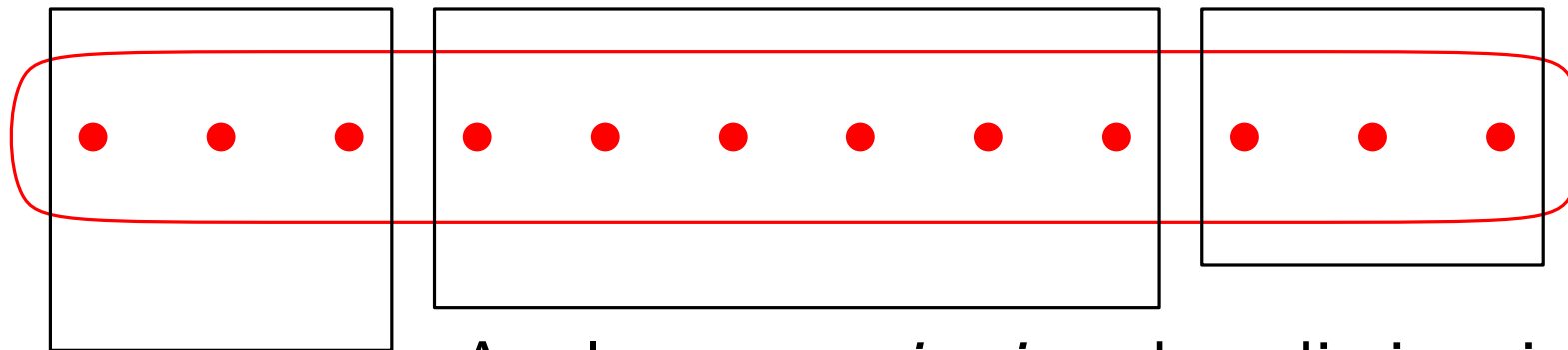
In  $O(n)$  time,  
or in  $O(n \log n)$  time.

[ Hopcroft and Wong 1974 ]

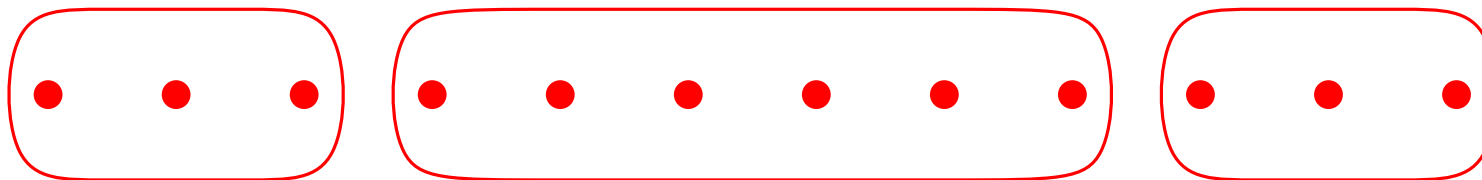
[ Hopcroft and Tarjan 1973 ]

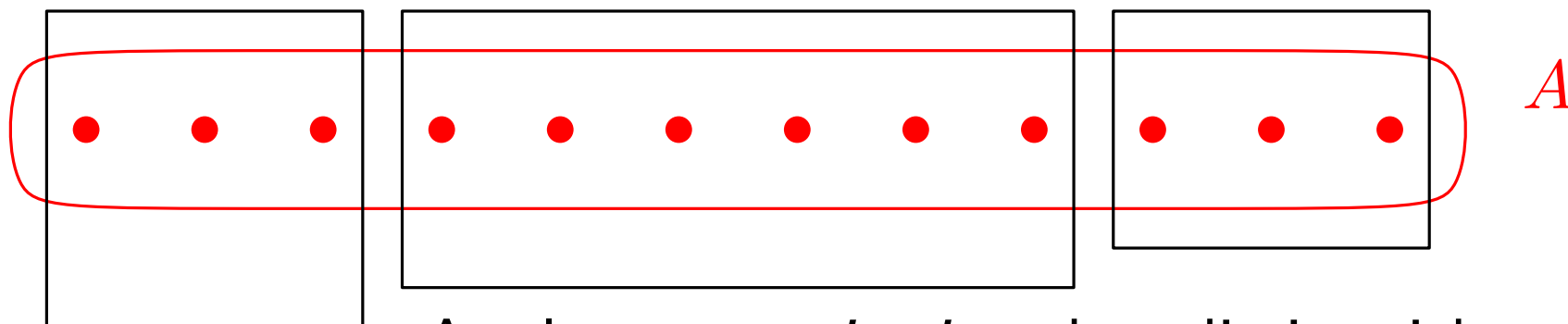


*A*

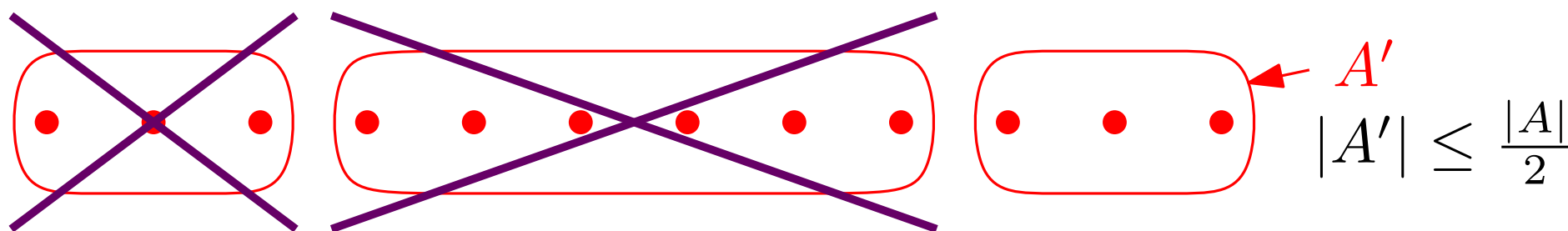


Apply some *criterion* that distinguishes points  
(distance from the center,  
number of closest neighbors, ...)



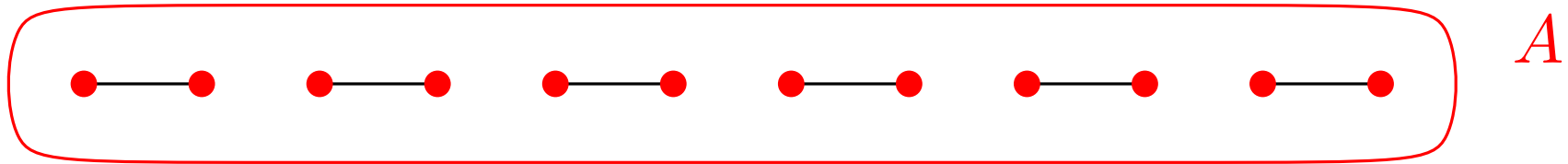


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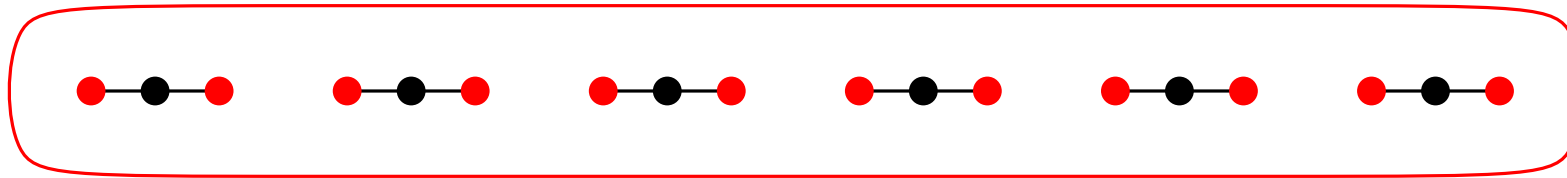
Throw away all but the smallest  
resulting class,  
and repeat.

Simultaneously apply this procedure to  $B$ .  
 $A'$  and  $B'$  may have *more* congruences!



Make some *construction*  
(midpoints of closest-pair edges, ...)

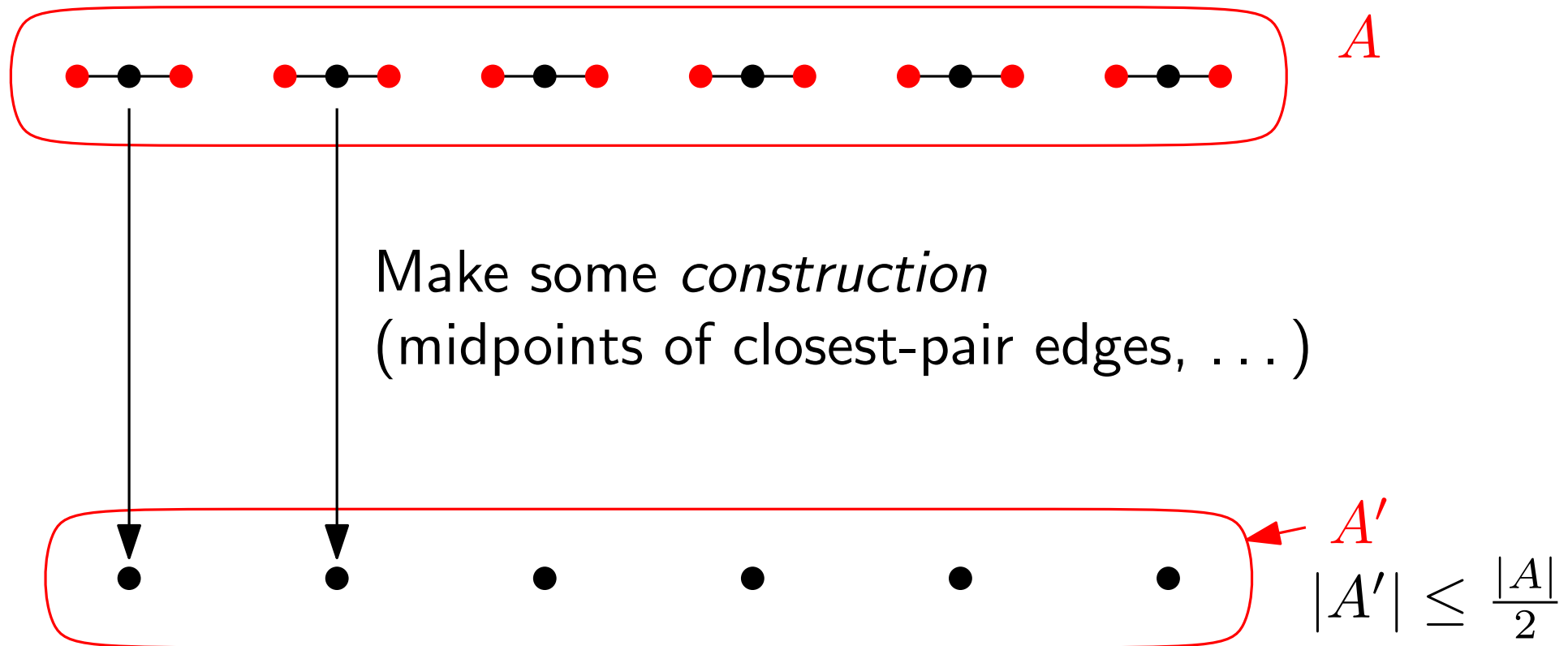
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*A*

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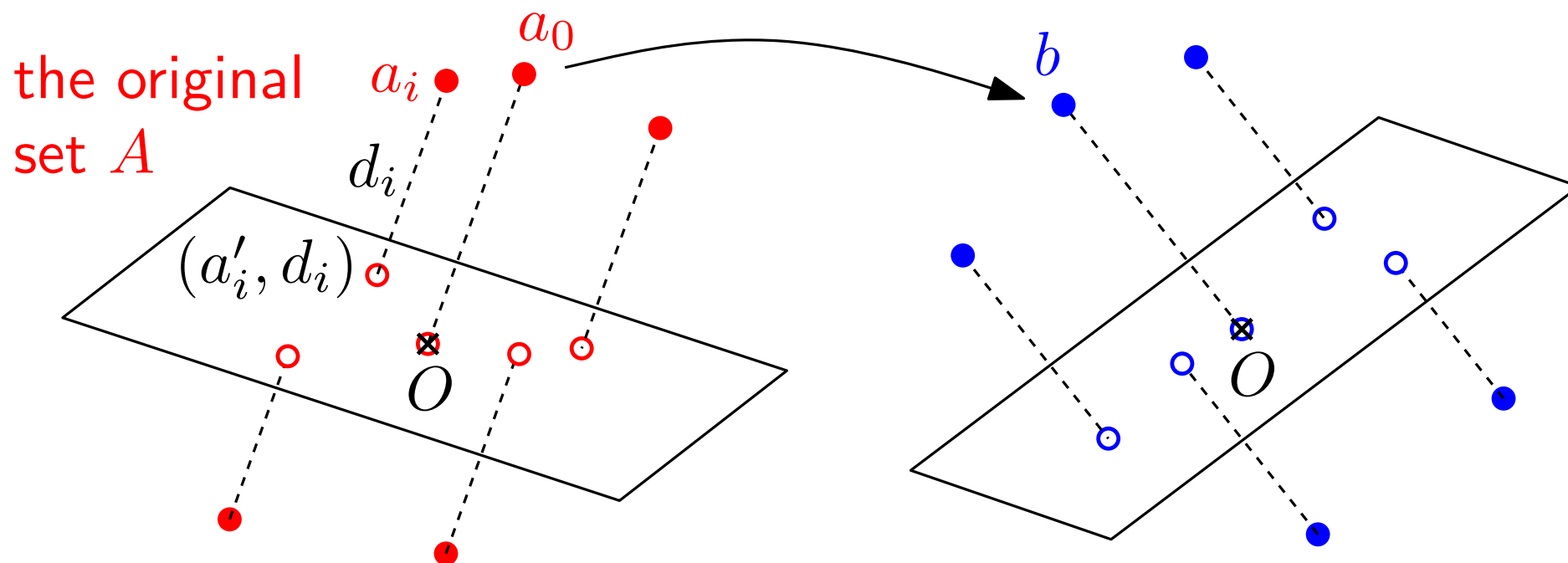


Simultaneously apply this procedure to  $B$ .  
 $A'$  and  $B'$  may have *more* congruences!

As soon as  $|A'| = |B'| = k$  is small:

Choose a point  $a_0 \in A'$  and try all  $k$  possibilities of mapping it to a point  $b \in B'$ .

Fixing  $a_0 \mapsto b$  reduces the dimension by one.



Project perpendicular to  $Oa_0$  and label projected points  $a'_i$  with the signed projection distance  $d_i$  as  $(a'_i, d_i)$ .

→ 2-dimensional congruence for LABELLED point sets

As soon as  $|A'| = |B'| = k$  is small:

Choose a point  $a_0 \in A'$  and try all  $k$  possibilities of mapping it to a point  $b \in B'$ .

Fixing  $a_0 \mapsto b$  reduces the dimension by one.

One problem in  $d$  dimensions is reduced to  $k$  problems in  $d - 1$  dimensions.

PRUNE by distance from the origin. If the points lie in a plane or on a line → DIMENSION REDUCTION.

Compute the convex hull.

If there are vertices of different degrees → PRUNE

The number  $n$  of vertices is reduced to  $\leq n/2$ . RESTART.

All  $n$  vertices have now degree 3, 4, or 5.

There are  $f = \frac{n}{2} + 2$  or  $f = n + 2$  or  $f = \frac{3n}{2} + 2$  faces.

If the face degrees are not all equal

→ switch to the centroids of the faces and PRUNE them.

$n$  is reduced to  $\leq \frac{3n}{4} + 1$ . RESTART.

Now  $P(A)$  must have the graph of a Platonic solid. →  $n \leq 20$ .  
→ DIMENSION REDUCTION.

# Three Dimensions [Akutsu 1995]

PRUNE by distance from the origin. If the points lie in a plane or on a line → DIMENSION REDUCTION.

Compute the convex hull. ←  $O(|A| \log |A|)$  time

If there are vertices of different degrees → PRUNE

The number  $n$  of vertices is reduced to  $\leq n/2$ . RESTART.

All  $n$  vertices have now degree 3, 4, or 5.

There are  $f = \frac{n}{2} + 2$  or  $f = n + 2$  or  $f = \frac{3n}{2} + 2$  faces.

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→ switch to the centroids of the faces and PRUNE them.

$n$  is reduced to  $\leq \frac{3n}{4} + 1$ . RESTART.

TIME =

$$O(n \log n) + O\left(\frac{3}{4}n \log \frac{3}{4}n\right) + O\left(\left(\frac{3}{4}\right)^2 n \log \left(\left(\frac{3}{4}\right)^2 n\right)\right) + \dots \\ = O(n \log n)$$

graph-theoretic pruning

PRUNE by distance from the origin. If the points lie in a plane or on a line → DIMENSION REDUCTION.

Compute the convex hull.

If there are vertices of different degrees → PRUNE

The number  $n$  of vertices is reduced to  $\leq n/2$ . RESTART.

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If the face degrees are not all equal

→ switch to the centroids of the faces and PRUNE them.

$n$  is reduced to  $\leq \frac{3n}{4} + 1$ . RESTART.

Now  $P(4)$  must have the graph of a Platonic solid. →  $n \leq 20$ .

→ DIMENSION REDUCTION.

PRUNE by distance from the origin. If the points lie in a plane or on a line  $\rightarrow$  DIMENSION REDUCTION.

Canonical point sets in 3d:

We get  $\leq 20$  two-dimensional projected point sets.

Rotate the plane to the  $x$ - $y$ -plane.

Compute the canonical 2-d point set.

$\leq 20$  candidates for canonical 3d point sets:

Choose the lex-smallest one.

Now  $P(4)$  must have the graph of a Platonic solid.  $\rightarrow n \leq 20$ .  
 $\rightarrow$  DIMENSION REDUCTION.

Function  $f(A) = A'$ ,  $A' \not\subseteq \{0\}$ , equivariant under rotations  $R$ :

$$f(RA) = RA'$$

$A'$  has *all symmetries* of  $A$  (and maybe more).

Primary goal:  $|A'| \leq |A| \cdot c$ ,  $c < 1$ .

If there is a chance, PRUNE and start from scratch with  $A'$  instead of  $A$ .

Ultimate goal:  $|A| \leq \text{const}$

Use some **more geometric** pruning to get:

Equivariant condensation on the 2-sphere:

Input:  $A \subseteq \mathbb{S}^2$ .

Output:  $A' \subseteq \mathbb{S}^2$ ,  $|A'| \leq \min\{|A|, 12\}$ ,  $A' = f(A)$  equivariant.

5 possibilities:

- $A' =$  vertices of a regular icosahedron
- $A' =$  vertices of a regular octahedron
- $A' =$  vertices of a regular tetrahedron
- $A' =$  two antipodal points, or
- $A' =$  a single point.

Dimension reduction without pruning:

Pick  $a_0 \in A$ . Try  $a_0 \mapsto b$  for all  $b \in B$  ( $n$  possibilities).

$\rightarrow O(n^{d-2} \log n)$  time [ Alt, Mehlhorn, Wagener, Welzl 1988 ]

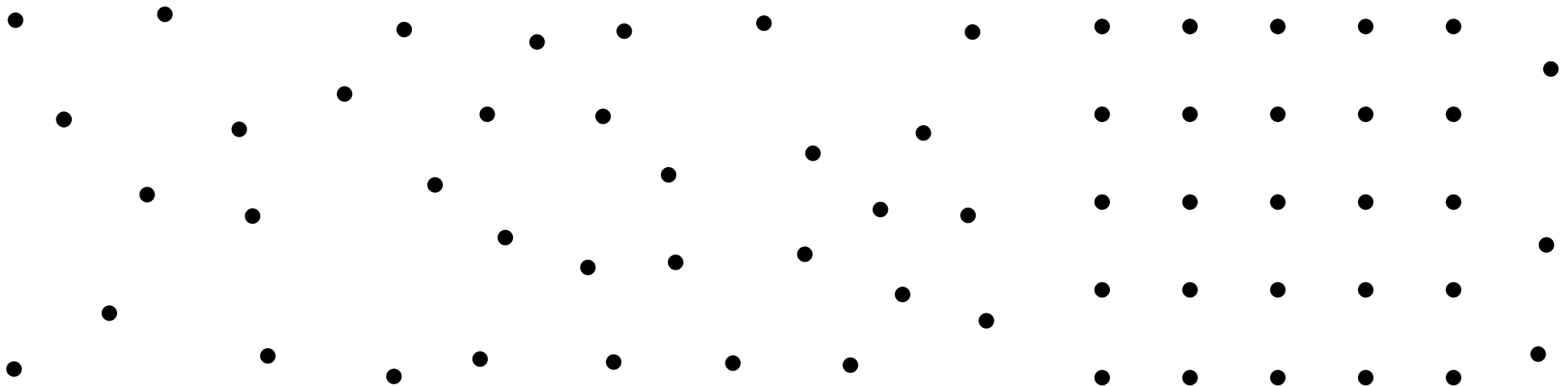
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**Closest pairs**  $(a, a')$ : [Matoušek  $\approx$  1998]

minimum distance  $\delta := \|a - a'\|$  among all pairs of vertices



Dimension reduction without pruning:

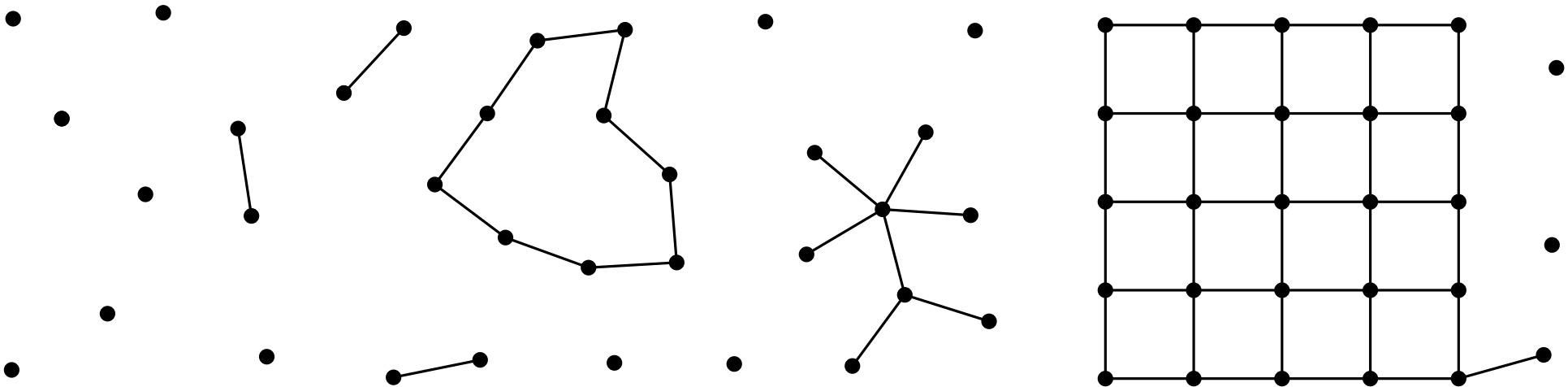
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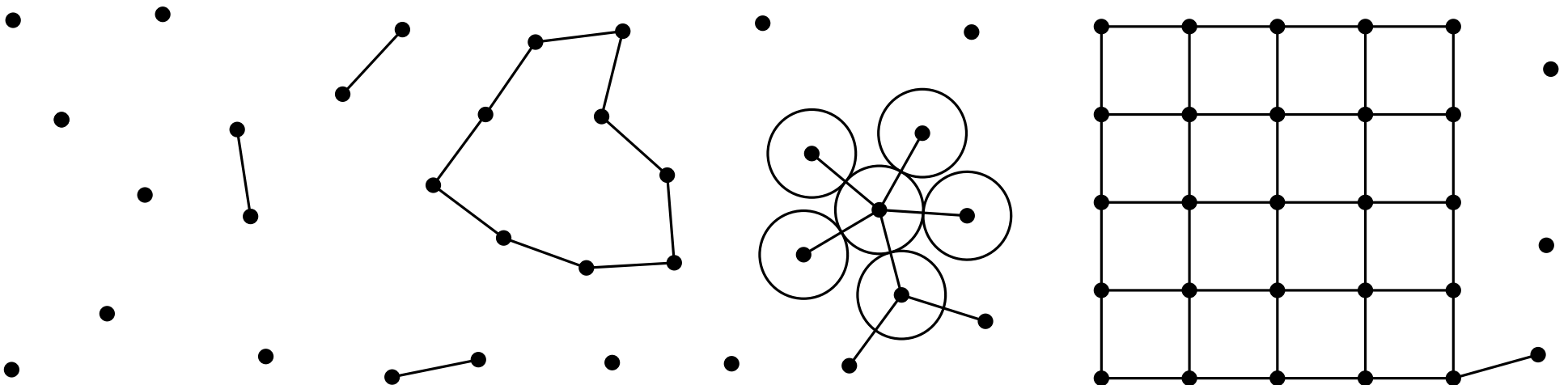
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**Closest pairs**  $(a, a')$ : [Matoušek  $\approx$  1998]

minimum distance  $\delta := \|a - a'\|$  among all pairs of vertices



Degree  $\leq$  the *kissing number*  $K_d$  (packing argument).

All closest pairs can be computed in  $O(n \log n)$  time ( $d$  fixed).

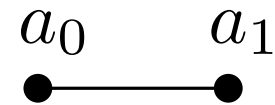
[ Bentley and Shamos, STOC 1976 ]

Dimension reduction without pruning:

Pick  $a_0 \in A$ . Try  $a_0 \mapsto b$  for all  $b \in B$  ( $n$  possibilities).

$\rightarrow O(n^{d-2} \log n)$  time [ Alt, Mehlhorn, Wagener, Welzl 1988 ]

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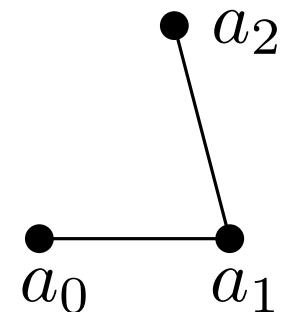
Pick a closest pair  $a_0 a_1$  in  $A$ . Try  $(a_0, a_1) \mapsto (b, b')$  for all closest pairs  $(b, b')$  in  $B$ .

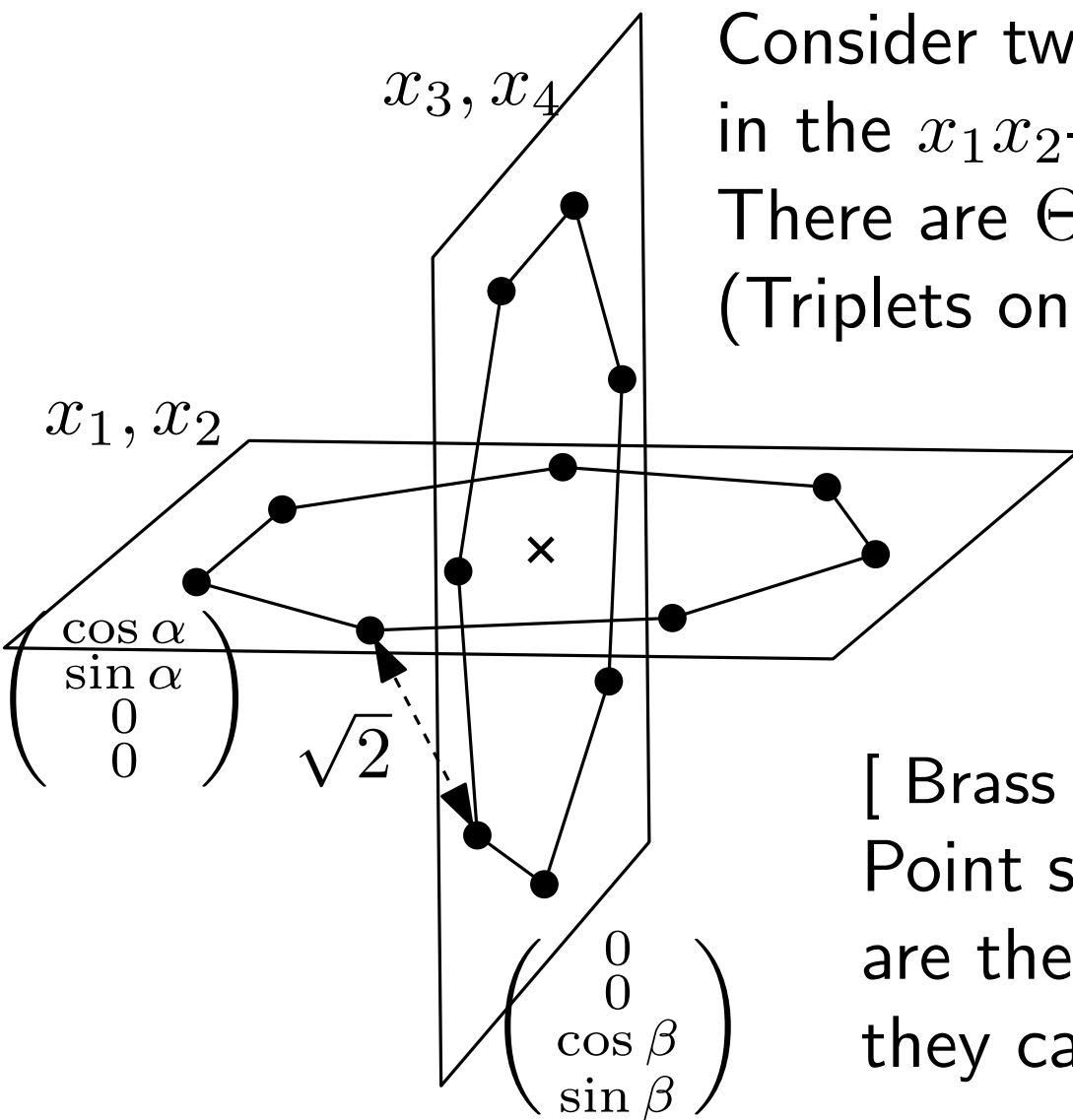
$O(n)$  possibilities, reducing the dimension by **two**.

$\rightarrow O(n^{\lfloor d/2 \rfloor} \log n)$  time [ Matoušek  $\approx$  1998 ]

---

Further improvement: Find a “closest triplet” ...





Consider two regular  $n$ -gons in the  $x_1x_2$ -plane and the  $x_3x_4$ -plane. There are  $\Theta(n^2)$  “closest triplets”. (Triplets on the same  $n$ -gon are not useful.)

The convex hull has  $\Theta(n^2)$  edges and facets.

[ Brass and Knauer 2002 ]  
Point sets in orthogonal subspaces are the only problematic case; they can be treated specially.

$\rightarrow O(n^{\lceil d/3 \rceil} \log n)$  time

- Random sampling
- Birthday paradox
- Closest pairs

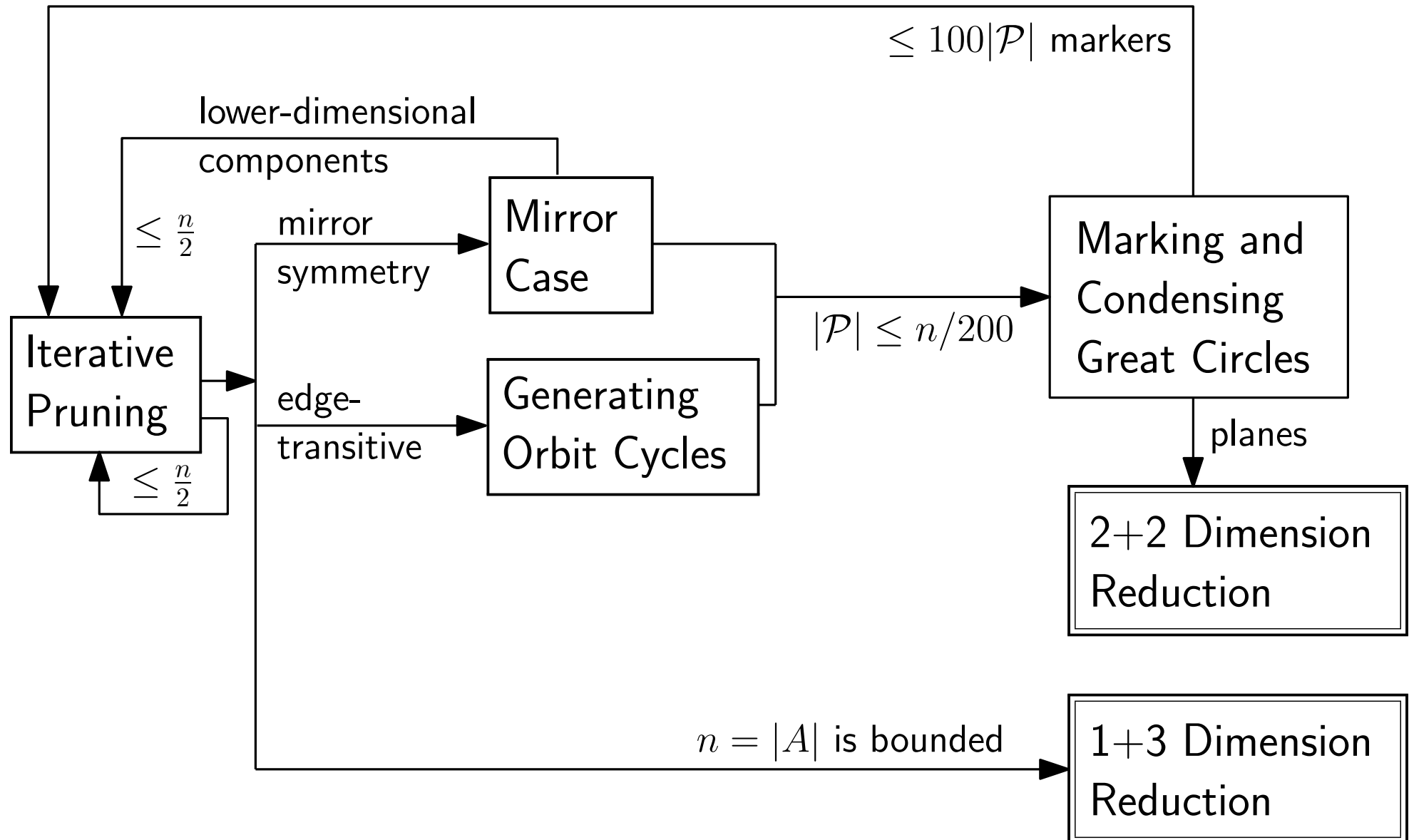
→ Monte Carlo algorithm,

$$O(n^{\lfloor d/2 \rfloor / 2} \log n) \text{ time, } O(n^{\lfloor d/2 \rfloor / 2}) \text{ space}$$

[Akutsu 1998 + improvement by J. Matoušek, personal communication]

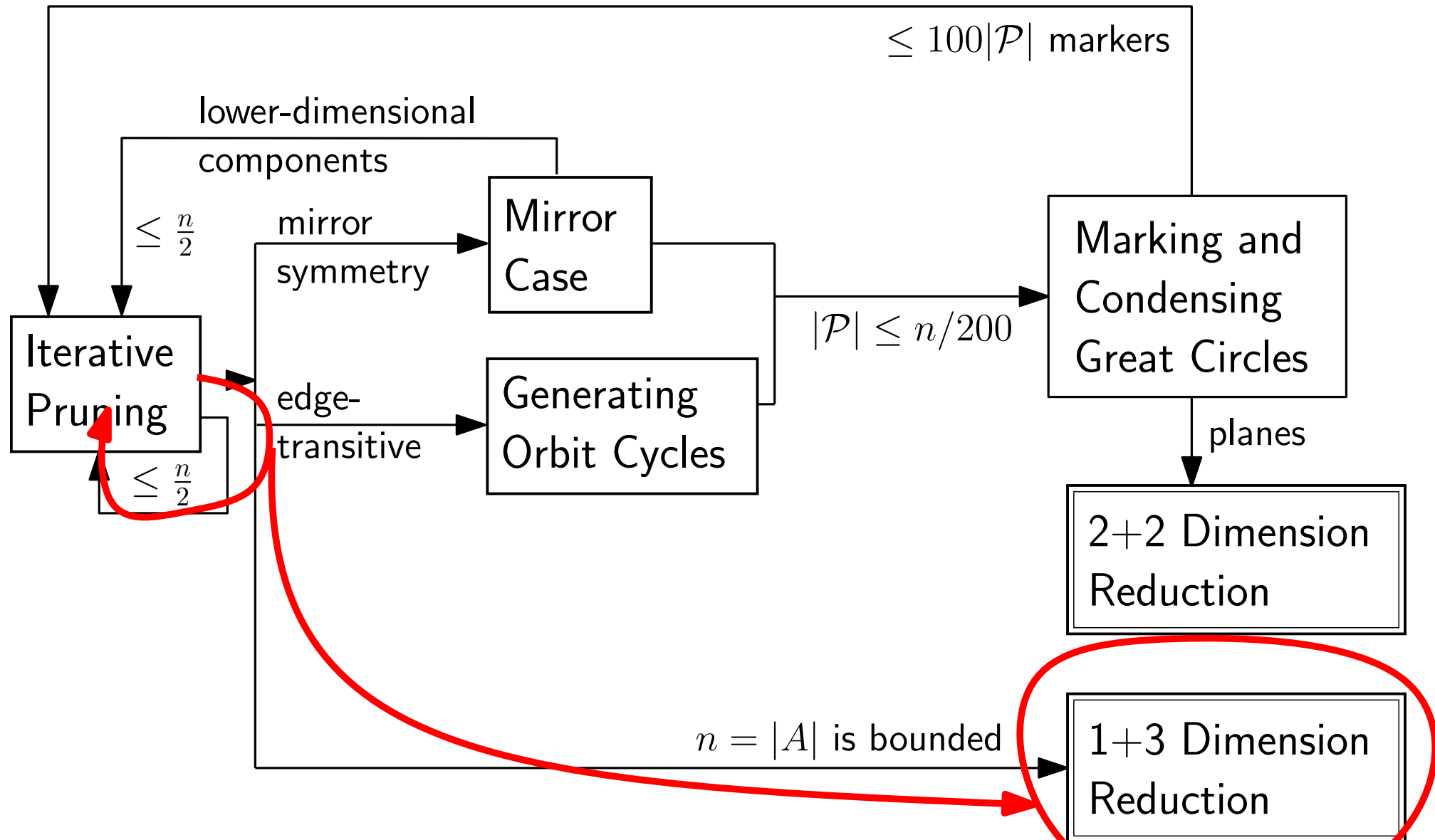
# 4 Dimensions: Algorithm Overview

joint work with Heuna Kim



# 4 Dimensions: Algorithm Overview

joint work with Heuna Kim



1) PRUNE by distance from the origin.

- $\implies$  we can assume that  $A$  lies on the 3-sphere  $\mathbb{S}^3$ .

2) Compute the closest pair graph

$$G(A) = (A, \{ uv : \|u - v\| = \delta \})$$

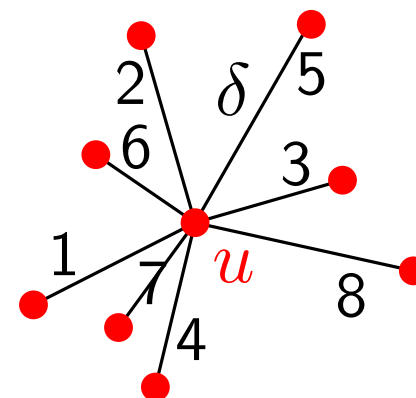
where  $\delta :=$  the distance of the closest pair, in  $O(n \log n)$  time.

- We can assume that  $\delta$  is SMALL:  $\delta \leq \delta_0 := 0.0005$ .  
(Otherwise,  $|A| \leq n_0$ , by a packing argument.)

# Everything Looks the Same!

By the PRUNING principle, we can assume that all points look locally the same:

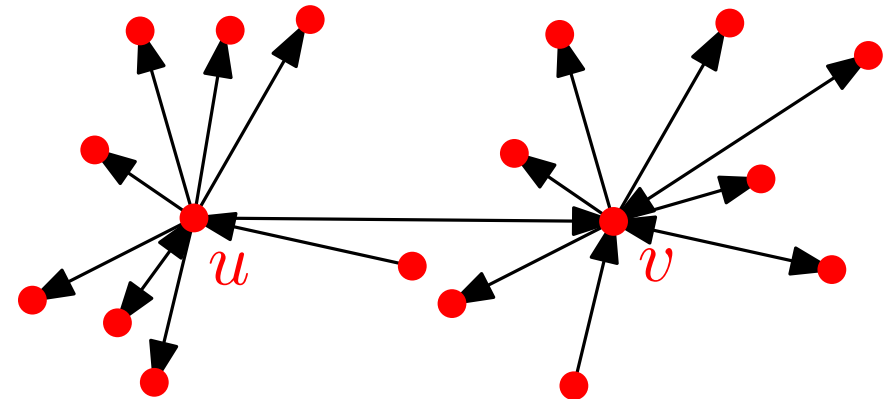
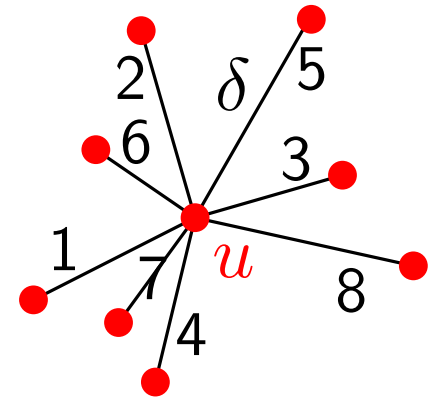
- All points have congruent neighborhoods in  $G(A)$ .  
(The neighbors of  $u$  lie on a 2-sphere in  $\mathbb{S}^3$ ;  
There are at most  $K_3 = 12$  neighbors.)



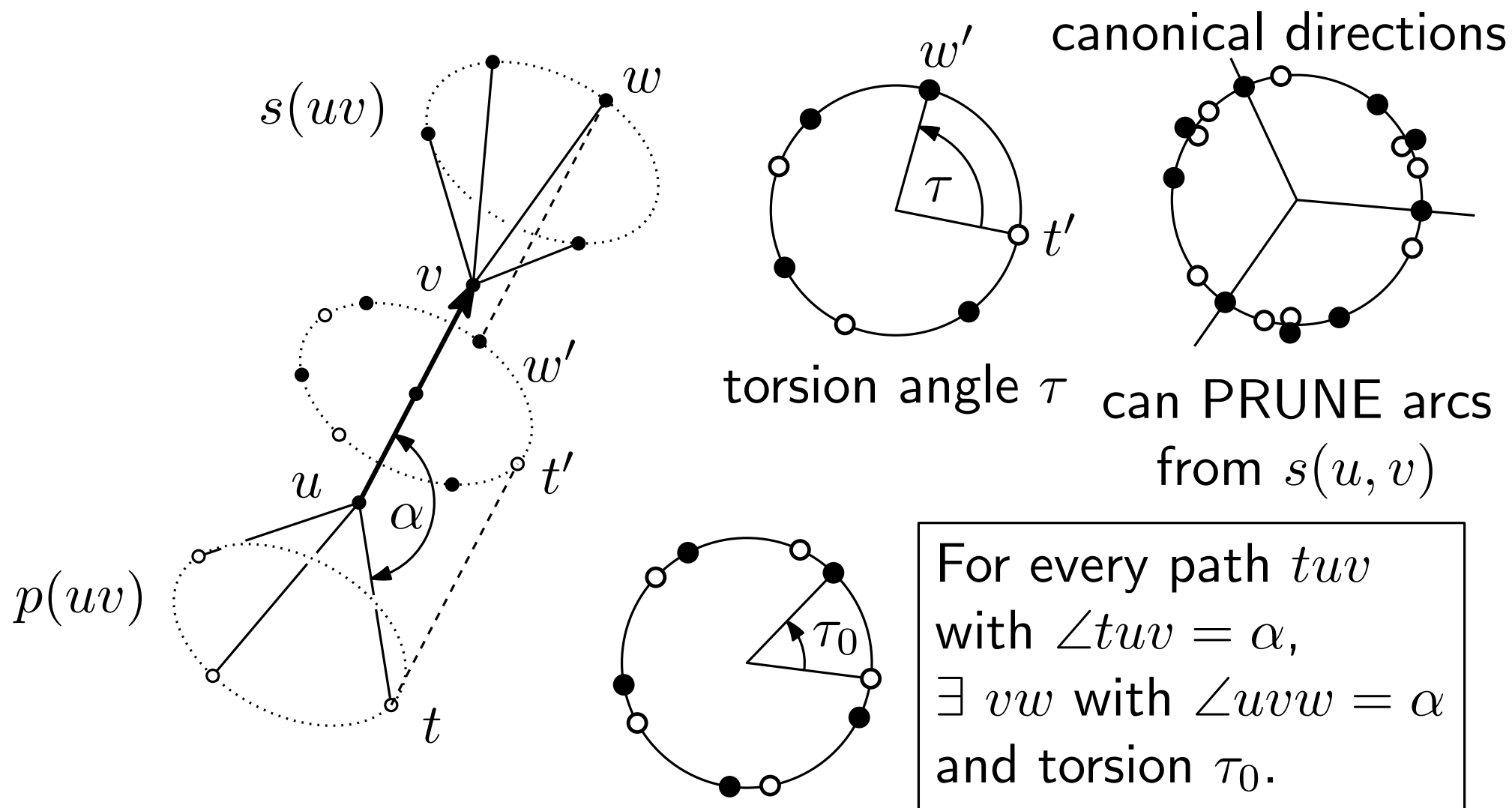
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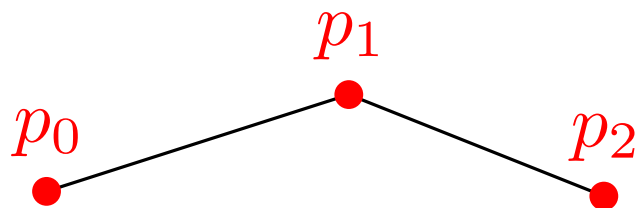
- All points have congruent neighborhoods in  $G(A)$ .  
(The neighbors of  $u$  lie on a 2-sphere in  $\mathbb{S}^3$ ;  
There are at most  $K_3 = 12$  neighbors.)
- Make a directed graph  $D$  from  $G(A)$   
and PRUNE its arcs  $uv$  by  
the **joint neighborhood** of  $u$  and  $v$ .



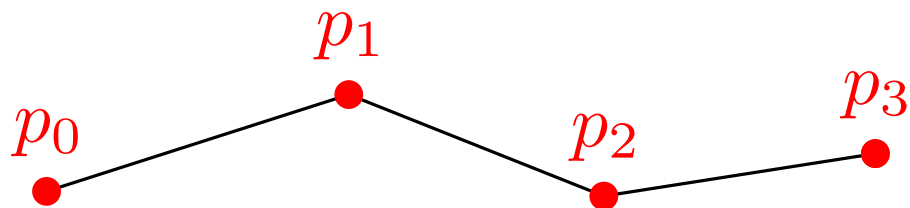
Pick some  $\alpha$ .  $s(uv) := \{vw : vw \in E, \angle uvw = \alpha\}$



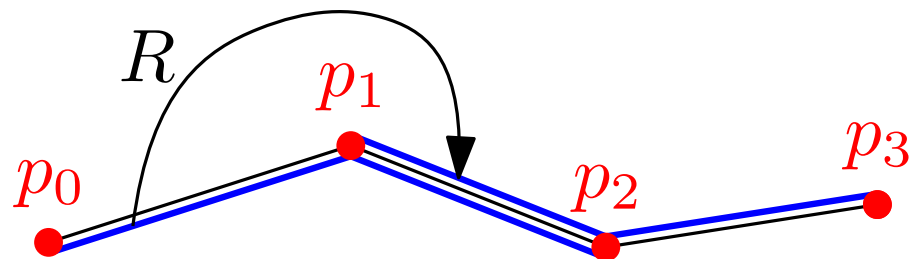
- ! For every path  $p_i p_{i+1} p_{i+2}$  with  $\angle p_i p_{i+1} p_{i+2} = \alpha$ ,  
•  $\exists p_{i+3}$  with  $\angle p_{i+1} p_{i+2} p_{i+3} = \alpha$  and torsion  $\tau_0$ .



- ! For every path  $p_i p_{i+1} p_{i+2}$  with  $\angle p_i p_{i+1} p_{i+2} = \alpha$ ,  
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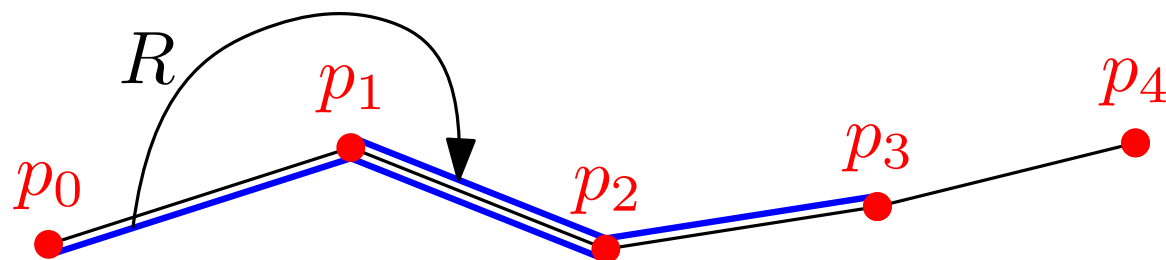


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$$R(p_0, p_1, p_2) = (p_1, p_2, p_3)$$

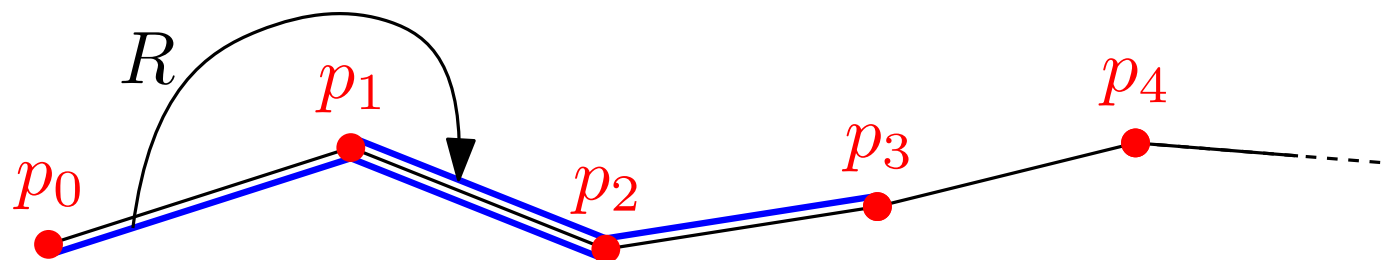
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$$R(p_0, p_1, p_2) = (p_1, p_2, p_3)$$

$$R(p_0, p_1, p_2, p_3) = (p_1, p_2, p_3, p_4)$$

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$$R(p_0, p_1, p_2, p_3) = (p_1, p_2, p_3, p_4)$$

$$R(p_1, p_2, p_3, p_4) = (p_2, p_3, p_4, p_5)$$

...

$Rp_i = p_{i+1}$ : The orbit of  $p_0$  under  $R$ , a helix

$$R = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \cos \psi & -\sin \psi \\ 0 & 0 & \sin \psi & \cos \psi \end{pmatrix} = \begin{pmatrix} R_\varphi & 0 \\ 0 & R_\psi \end{pmatrix}$$

in some appropriate coordinate system.

$\varphi \neq \pm\psi$ :  $\rightarrow$  unique decomposition  $\mathbb{R}^4 = P \oplus Q$  into two completely orthogonal 2-dimensional *axis planes*  $P$  and  $Q$

$\varphi = \pm\psi$ : isoclinic rotations

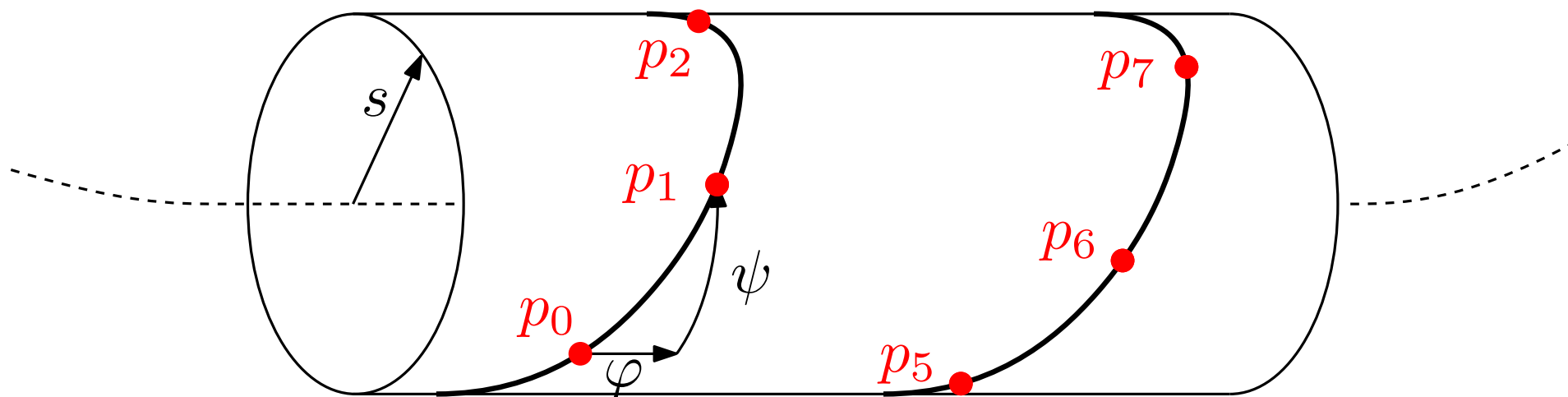
The orbit of a point  $a_0 = (x_1, x_2, x_3, x_4)$  lies on a *helix*  
on a *flat torus*  $C_r \times C_s$ , with  $r = \sqrt{x_1^2 + x_2^2}$ ,  $s = \sqrt{x_3^2 + x_4^2}$



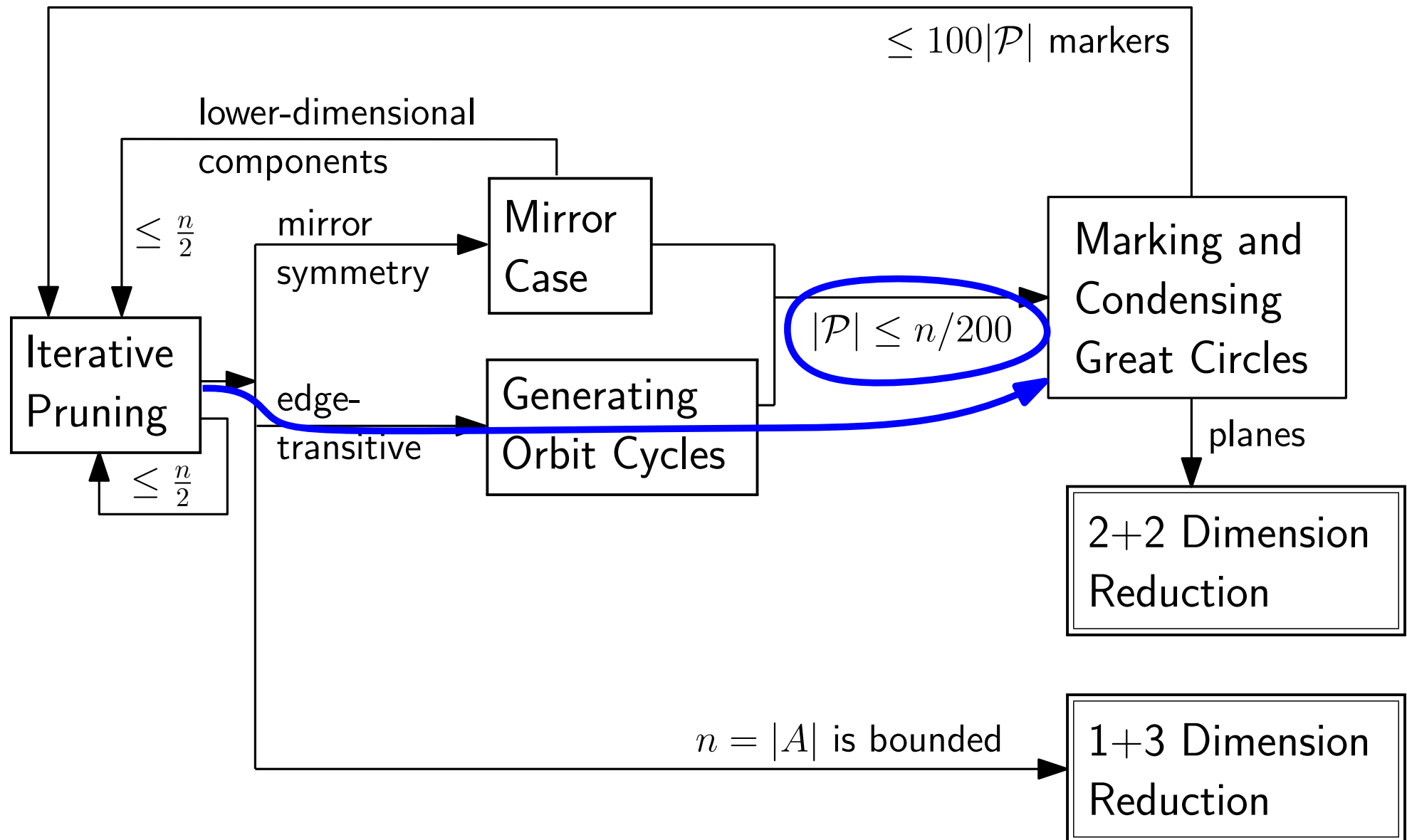
circle with radius  $r$

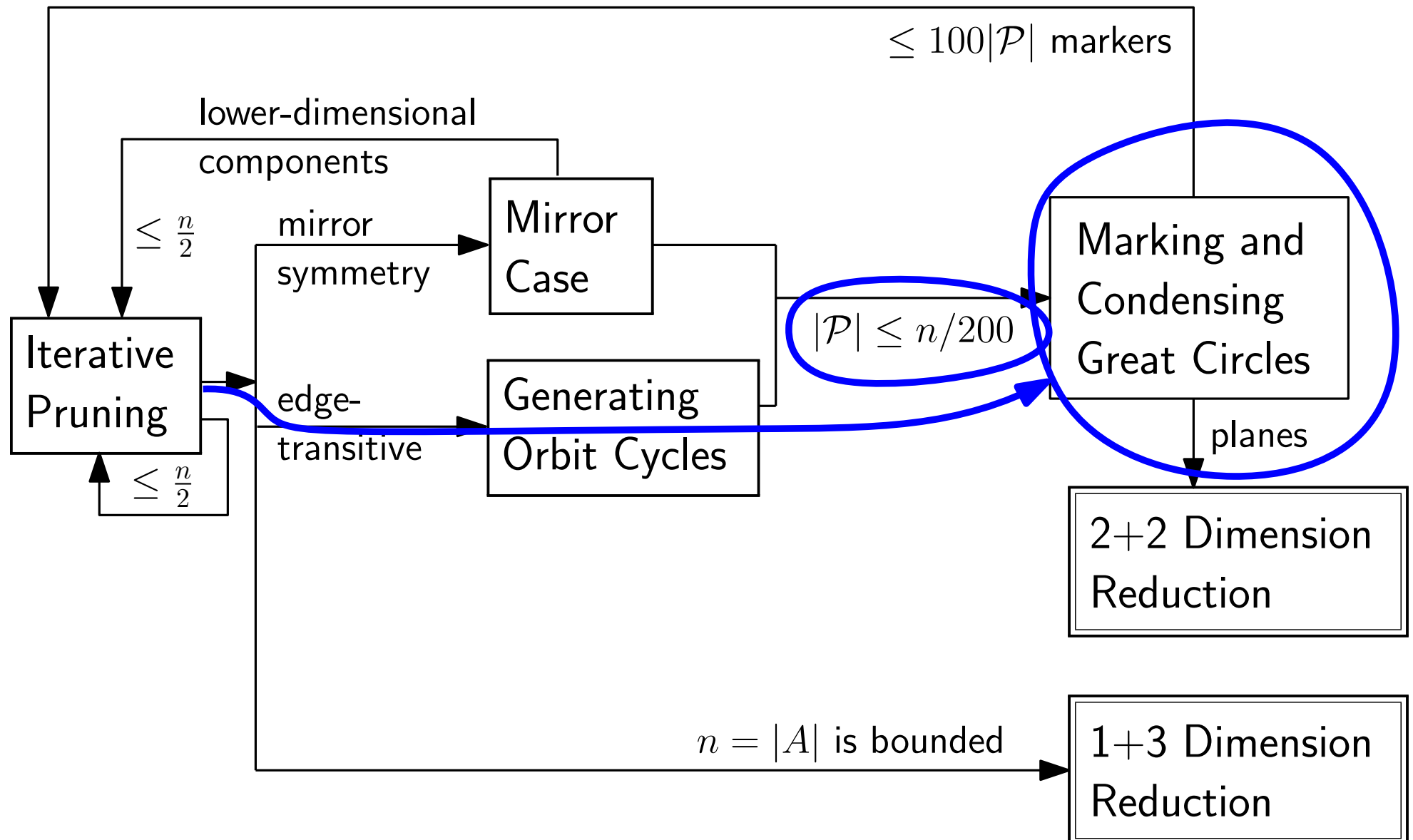
$$R = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \cos \psi & -\sin \psi \\ 0 & 0 & \sin \psi & \cos \psi \end{pmatrix} = \begin{pmatrix} R_\varphi & 0 \\ 0 & R_\psi \end{pmatrix}$$

The orbit of a point  $p_0 = (x_1, y_1, x_2, y_2)$  lies on a *helix* on a *flat torus*  $C_r \times C_s$ , with  $r = \sqrt{x_1^2 + y_1^2}$ ,  $s = \sqrt{x_2^2 + y_2^2}$

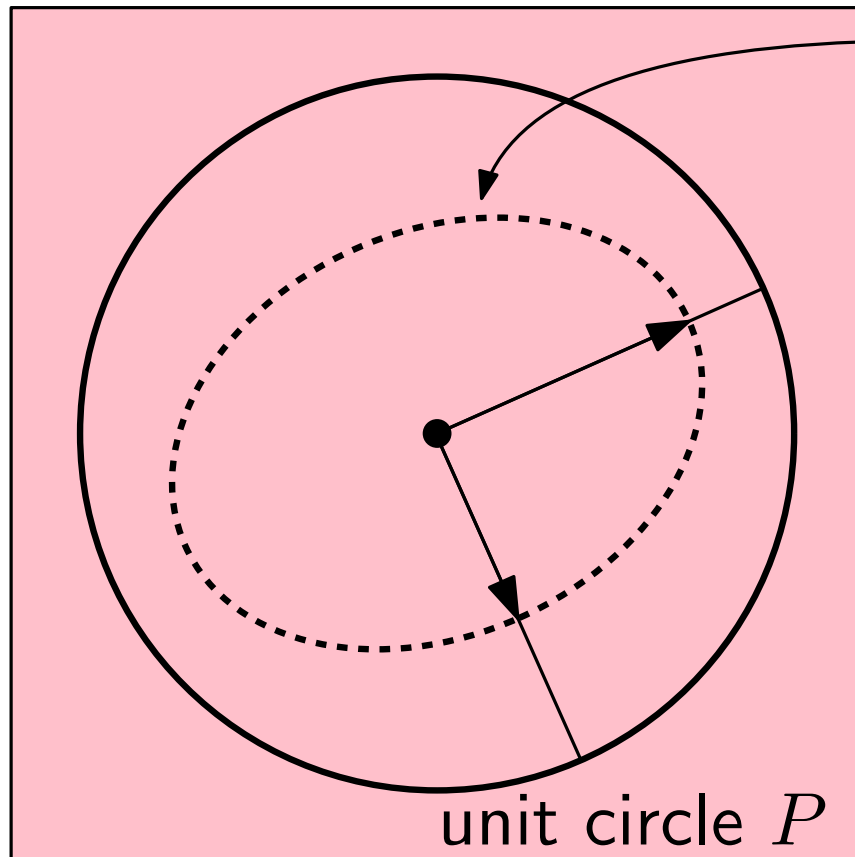


- Every point lies on  $\leq 60$  orbit cycles.
  - Every orbit cycle contains  $\geq 12000$  points, because  $\delta$  is small.
  - Every orbit cycle generates 1 plane (corresponding to the smaller of  $\varphi$  and  $\psi$ .)
- $\implies$  a collection of  $\leq n/200$  planes (or: great circles)



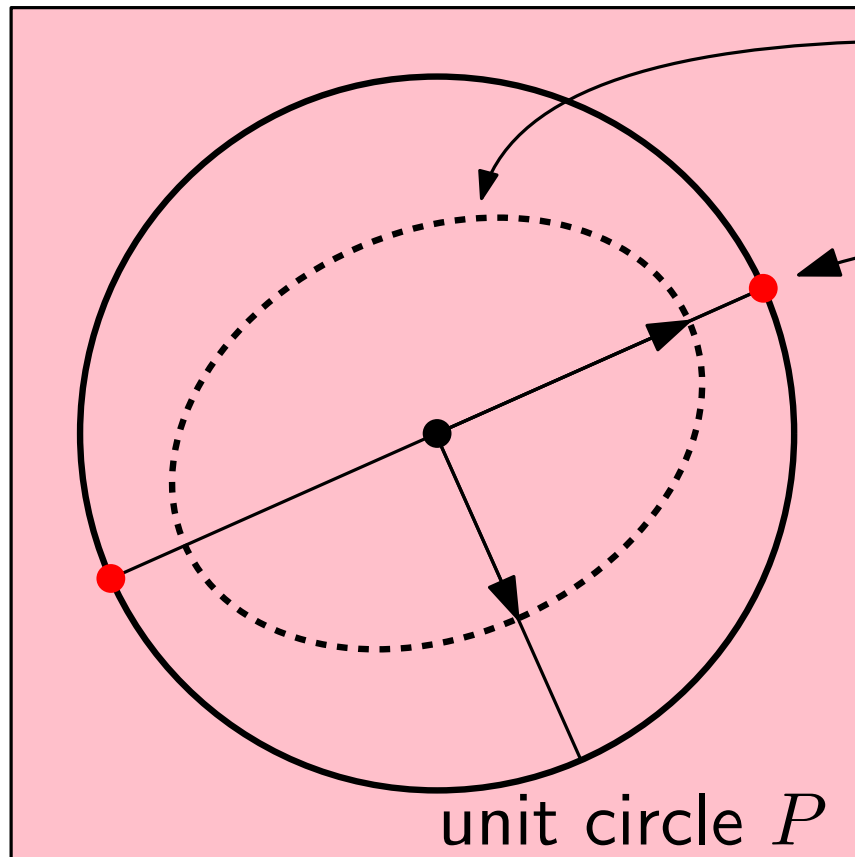


# Marking Points on Great Circles



projection of another unit circle  $Q$

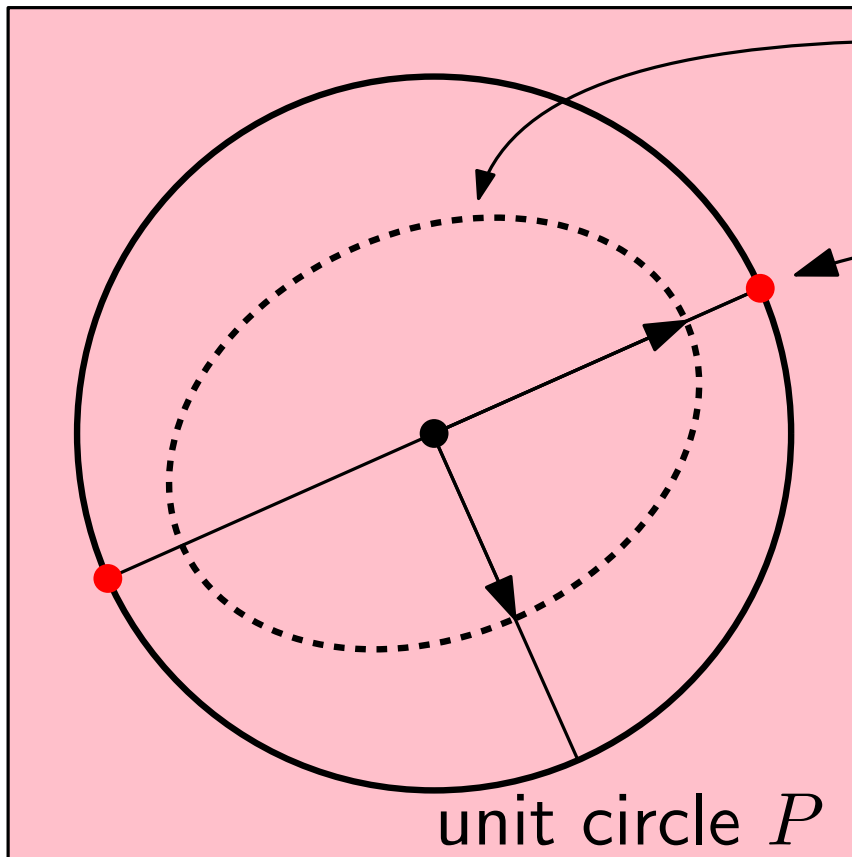
# Marking Points on Great Circles



projection of another unit circle  $Q$

IDEA: mark those two points in  $P$

# Marking Points on Great Circles



projection of ~~another unit circle~~  $Q$   
a neighbor of  $P$

IDEA: mark those two points in  $P$

IDEA 2: Construct the closest-pair graph in the space of great circles, in  $O(n \log n)$  time.

Every plane has at most  $K_5 \leq 44$  neighbors.

planes in 4-space  $\Leftrightarrow$  great circles on  $\mathbb{S}^3 \Leftrightarrow$  a.k.a. lines in  $\mathbb{R}P^3$

plane through  $(x_1, y_1, x_2, y_2)$  and  $(x'_1, y'_1, x'_2, y'_2)$  :

$$(v_1, \dots, v_6) = \left( \begin{vmatrix} x_1 & y_1 \\ x'_1 & y'_1 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ x'_1 & x'_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_2 \\ x'_1 & y'_2 \end{vmatrix}, \begin{vmatrix} y_1 & x_2 \\ y'_1 & x'_2 \end{vmatrix}, \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}, \begin{vmatrix} x_2 & y_2 \\ x'_2 & y'_2 \end{vmatrix} \right)$$

$(v_1, \dots, v_6) \in \mathbb{R}P^5$ . [Plücker relations  $v_1v_6 - v_2v_5 + v_3v_4 = 0$ ]

Normalize:

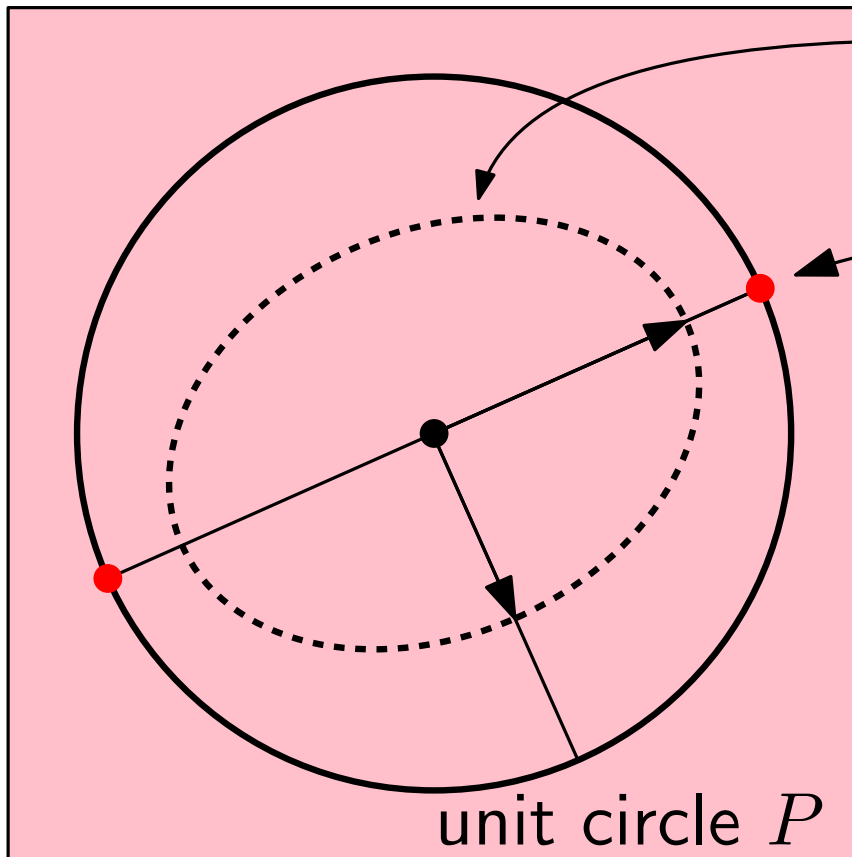
$\rightarrow$  A great circle is represented by two antipodal points on  $\mathbb{S}^5$ .

This representation is geometrically meaningful:

Distances on  $\mathbb{S}^5$  are preserved under rotations of  $\mathbb{R}^4 / \mathbb{S}^3$ .

(Packings of 2-planes in 4-space were considered by [Conway, Hardin and Sloan 1996], with different distances.)

# Marking Points on Great Circles

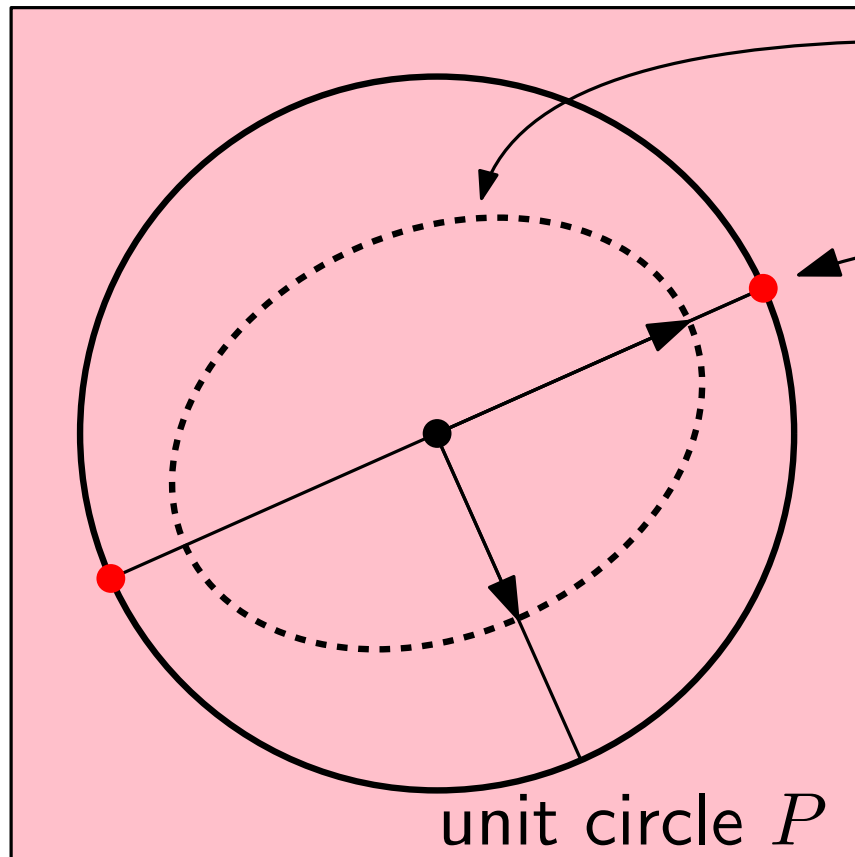


projection of ~~another unit circle~~  $Q$   
a neighbor of  $P$

IDEA: mark those two points in  $P$

IDEA 2: Construct the closest-pair graph in the space of great circles, in  $O(n \log n)$  time.

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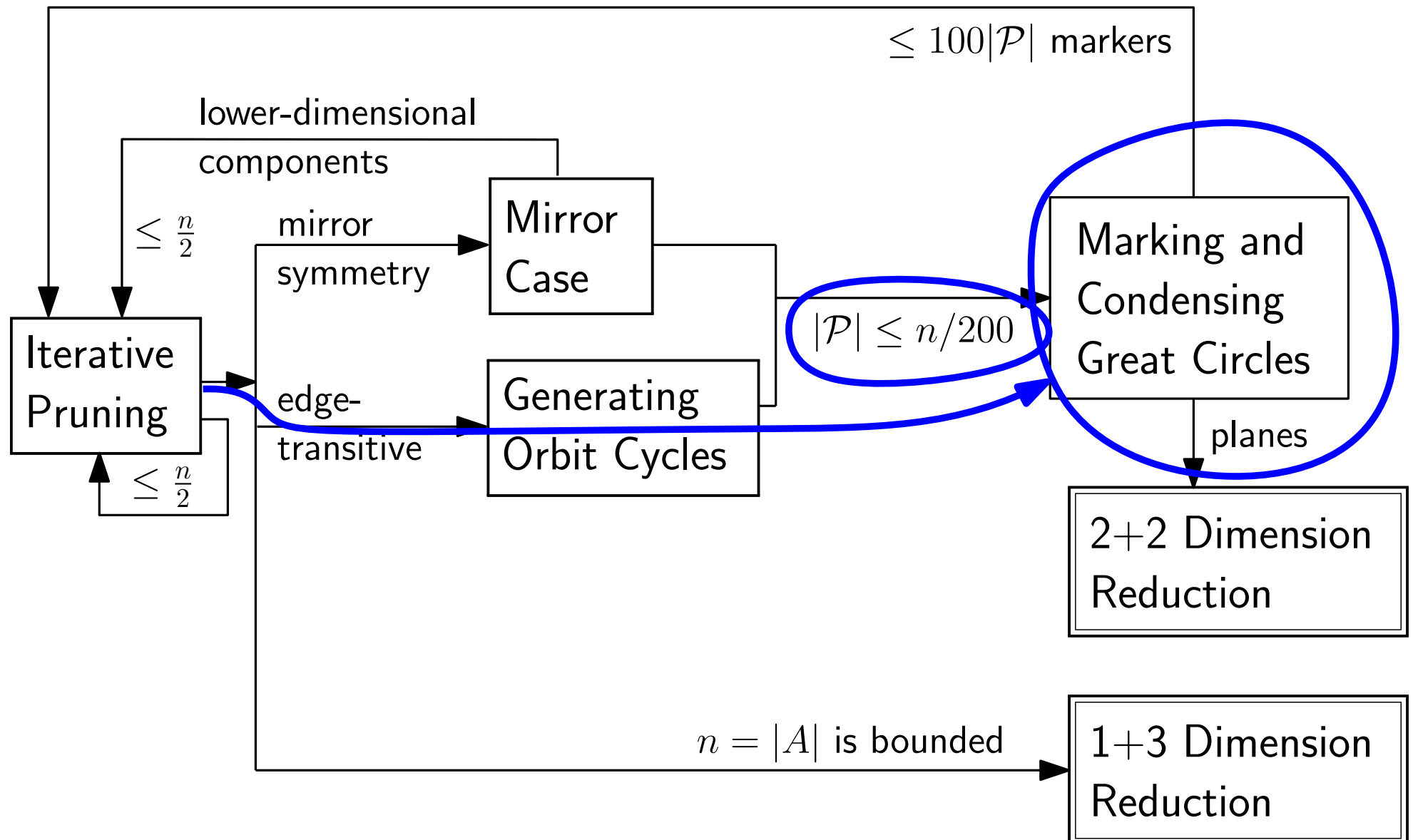
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$m \leq \frac{n}{200}$  great circles in  $\mathbb{R}^4 \longrightarrow m$  point pairs on  $\mathbb{S}^5$

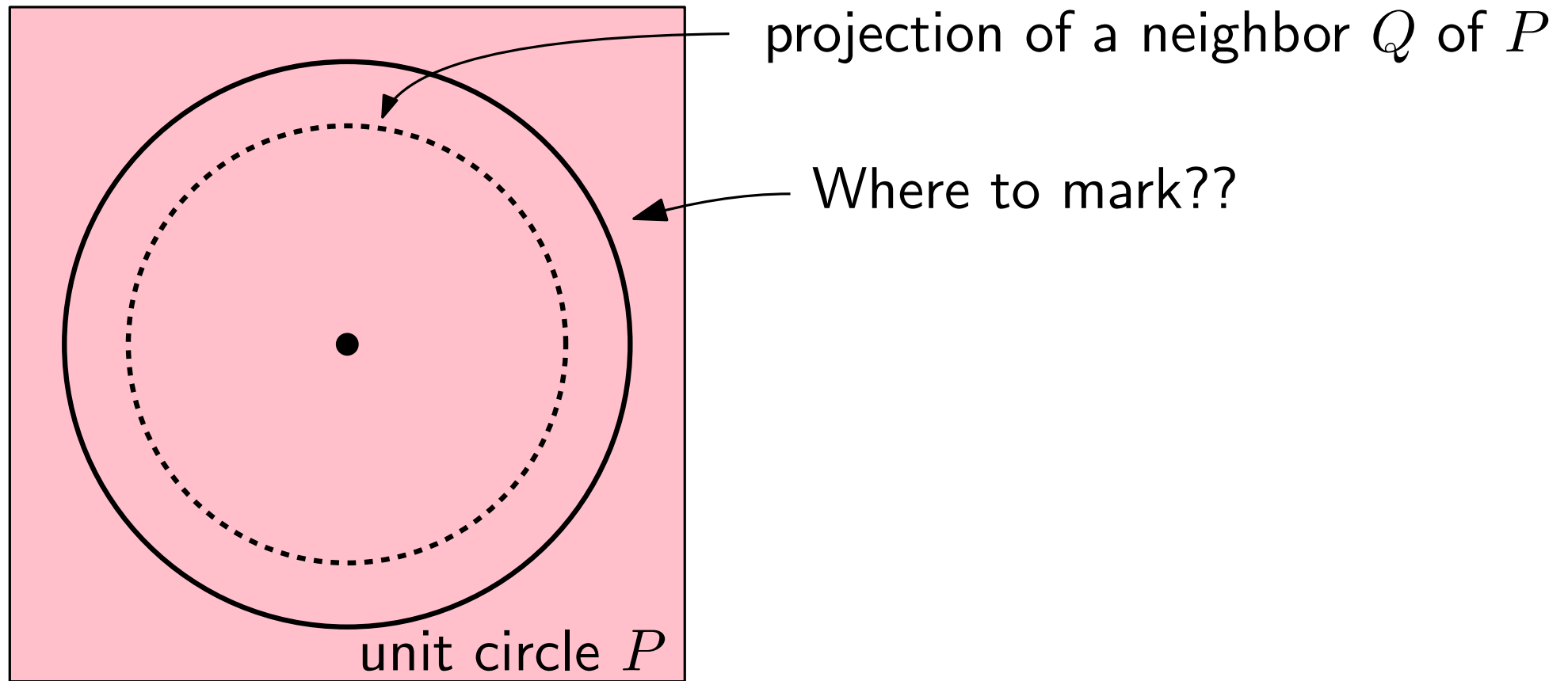
At most 88 points are marked on every great circle.

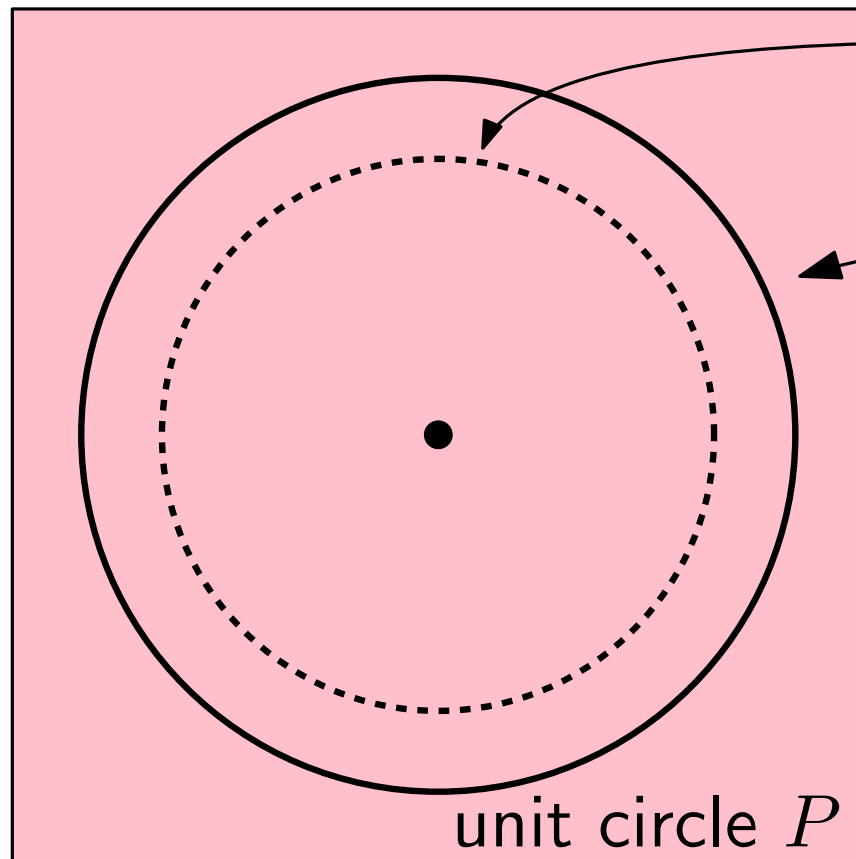
These points replace  $A$ .  $\rightarrow$  successful PRUNING



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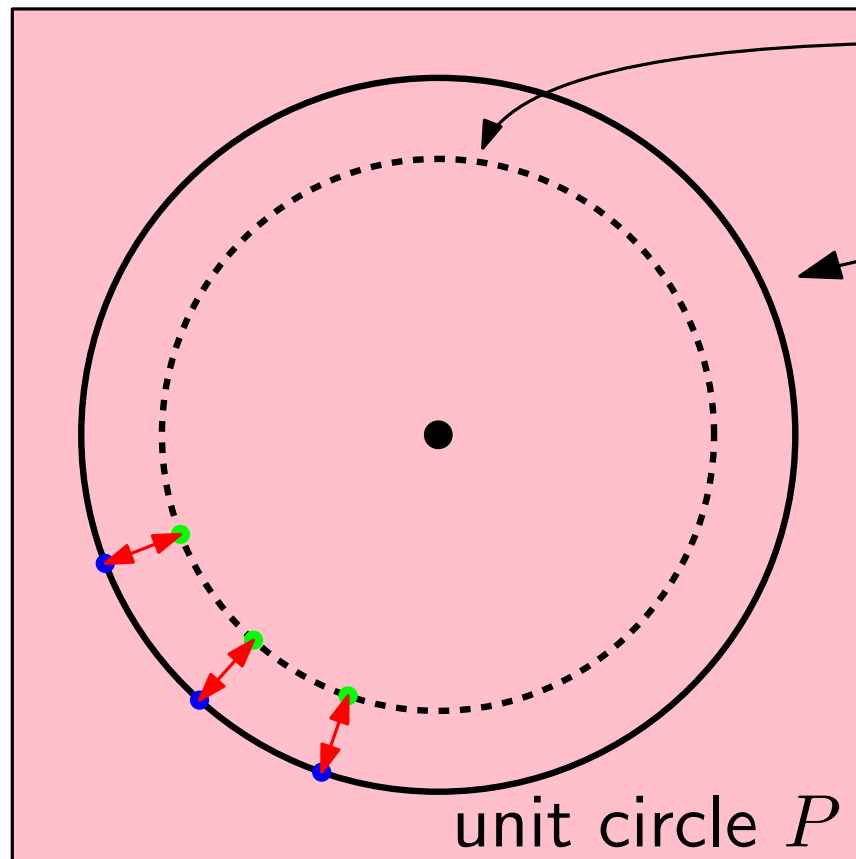




projection of a neighbor  $Q$  of  $P$

Where to mark??

Problem if *all* closest pairs are isoclinic.



projection of a neighbor  $Q$  of  $P$

Where to mark??

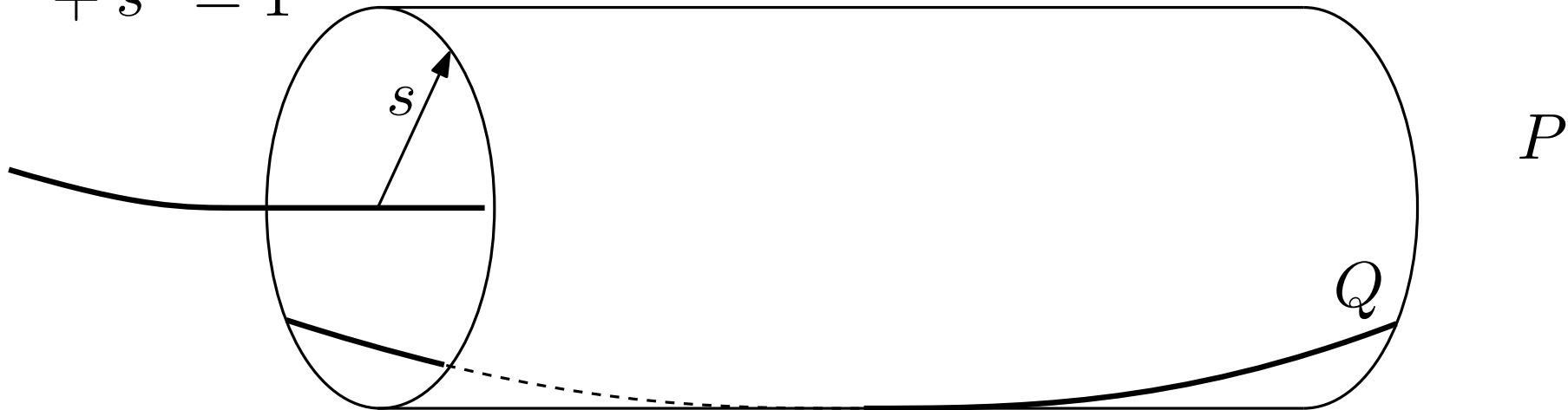
Problem if *all* closest pairs are isoclinic.

Constant distances from one circle to the other.

“Clifford-parallel”  $\equiv$  isoclinic

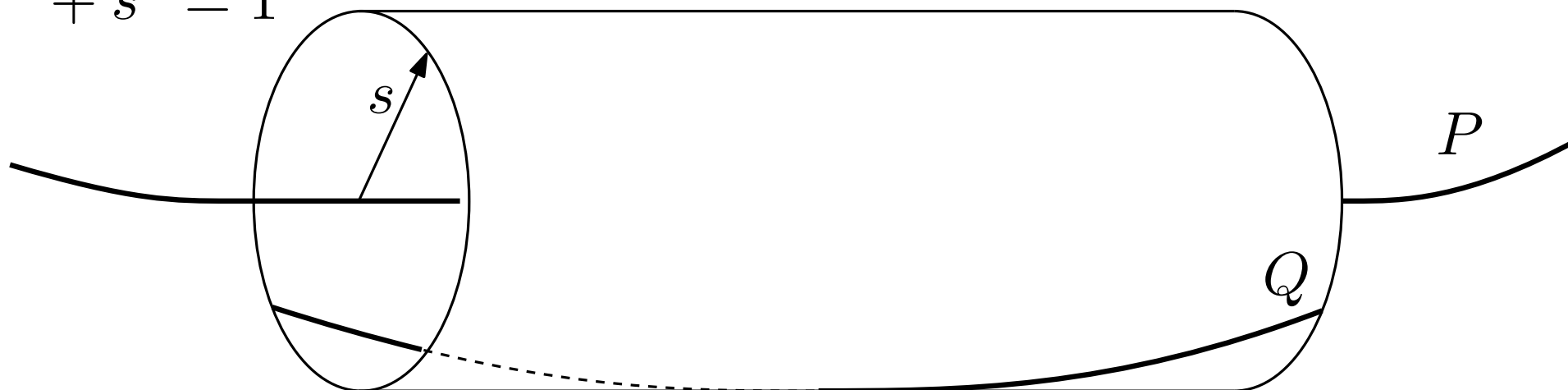
$$P: \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix}, \quad Q: \begin{pmatrix} r \cos t \\ r \sin t \\ s \cos(\alpha + t) \\ s \sin(\alpha + t) \end{pmatrix}$$

$$r^2 + s^2 = 1$$



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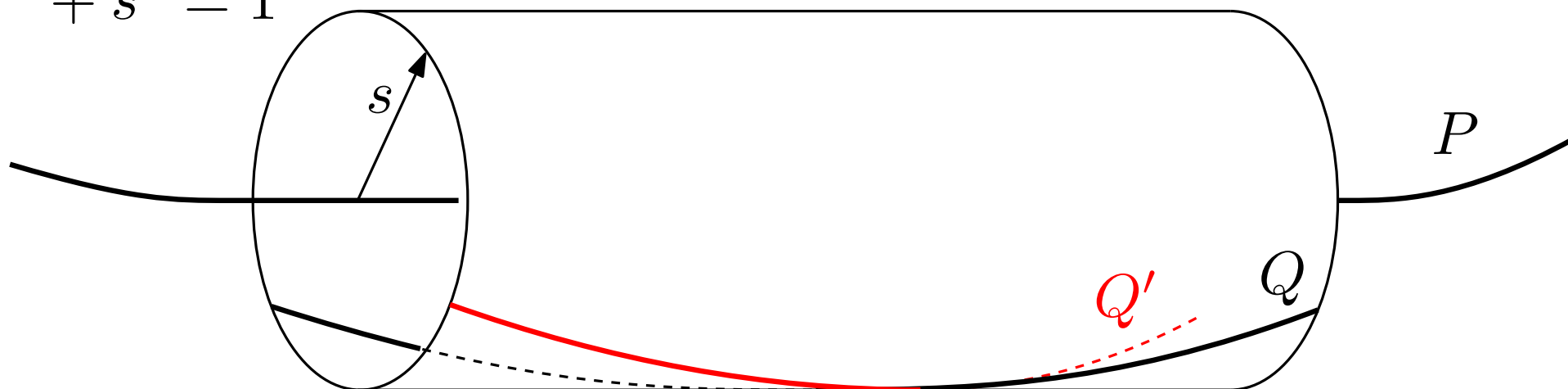
$$h(x_1, y_1, x_2, y_2) = \text{the right Hopf map } h: \mathbb{S}^3 \rightarrow \mathbb{S}^2$$

$$\left( 2(x_1 y_2 - y_1 x_2), 2(x_1 x_2 + y_1 y_2), 1 - 2(x_2^2 + y_2^2) \right)$$

[ Hopf 1931 ]

$$P: \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix}, \quad Q: \begin{pmatrix} r \cos t \\ r \sin t \\ s \cos(\alpha + t) \\ s \sin(\alpha + t) \end{pmatrix} \quad Q': \begin{pmatrix} r \cos t \\ r \sin t \\ s \cos(\alpha - t) \\ s \sin(\alpha - t) \end{pmatrix}$$

$$r^2 + s^2 = 1$$



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[ Hopf 1931 ]

Right Hopf map  $h: \mathbb{S}^3 \rightarrow \mathbb{S}^2$

The fibers  $h^{-1}(p)$  for  $p \in \mathbb{S}^2$  are great circles: a *Hopf bundle*

Every great circle belongs to a unique right Hopf bundle.

Isoclinic  $\equiv$  belong to the same Hopf bundle

This is a transitive relation.

Right Hopf map  $h: \mathbb{S}^3 \rightarrow \mathbb{S}^2$

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Isoclinic  $\equiv$  belong to the same Hopf bundle

This is a transitive relation.

If all closest pairs are isoclinic

→ all great circles in a connected component of the closest-pair graph belong to the same bundle.

→  $h$  maps them to points on  $\mathbb{S}^2$ .

We know how to deal with  $\mathbb{S}^2$ !

## Equivariant condensation on the 2-sphere:

Input:  $A \subseteq \mathbb{S}^2$ .

Output:  $A' \subseteq \mathbb{S}^2$ ,  $|A'| \leq \min\{|A|, 12\}$ .

- $A'$  = vertices of a regular icosahedron
- $A'$  = vertices of a regular octahedron
- $A'$  = vertices of a regular tetrahedron
- $A'$  = two antipodal points, or
- $A'$  = a single point.

Equivariant condensation on the 2-sphere:

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- $A' =$  a single point.

Condense each connected component of the closest-pair graph to  $\leq 12$  great circles.

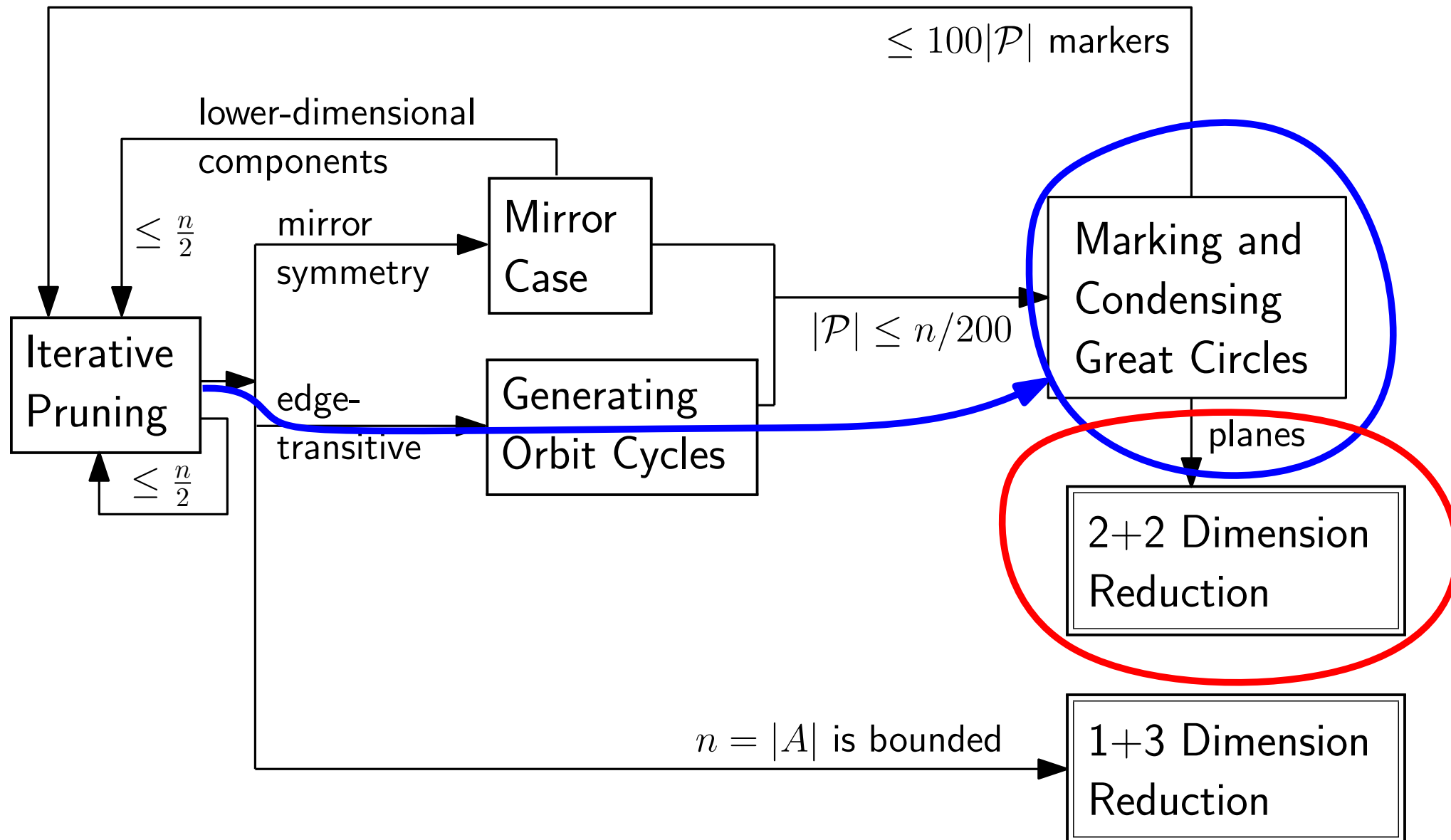
Compute closest-pair graph (on  $\mathbb{S}^5$ ) from scratch.

If no progress, distance between closest pairs is  $\geq D_{\text{icosa}}$

$\rightarrow \leq 829$  great circles  $\rightarrow$  2+2 DIMENSION REDUCTION

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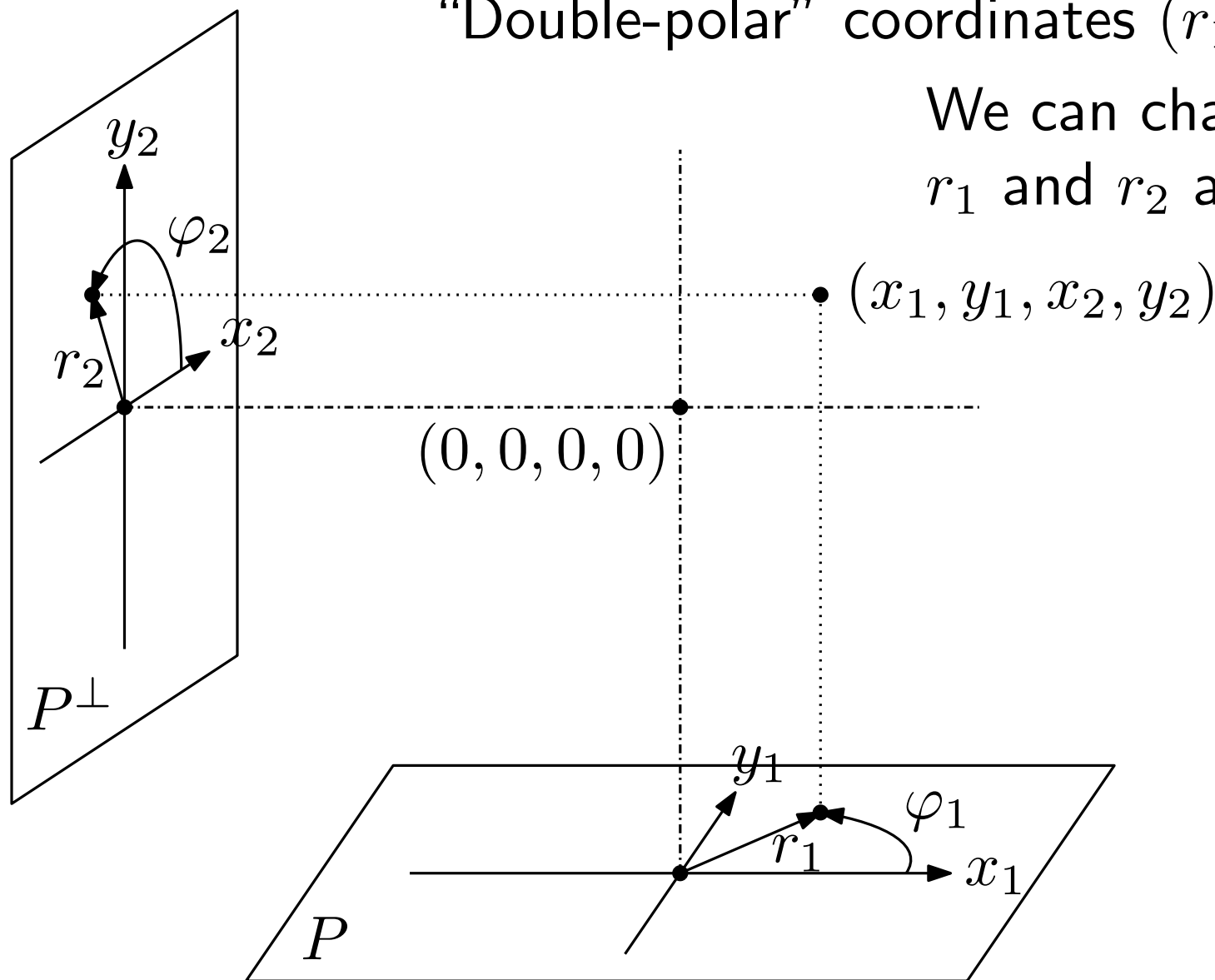


# 2+2 Dimension Reduction

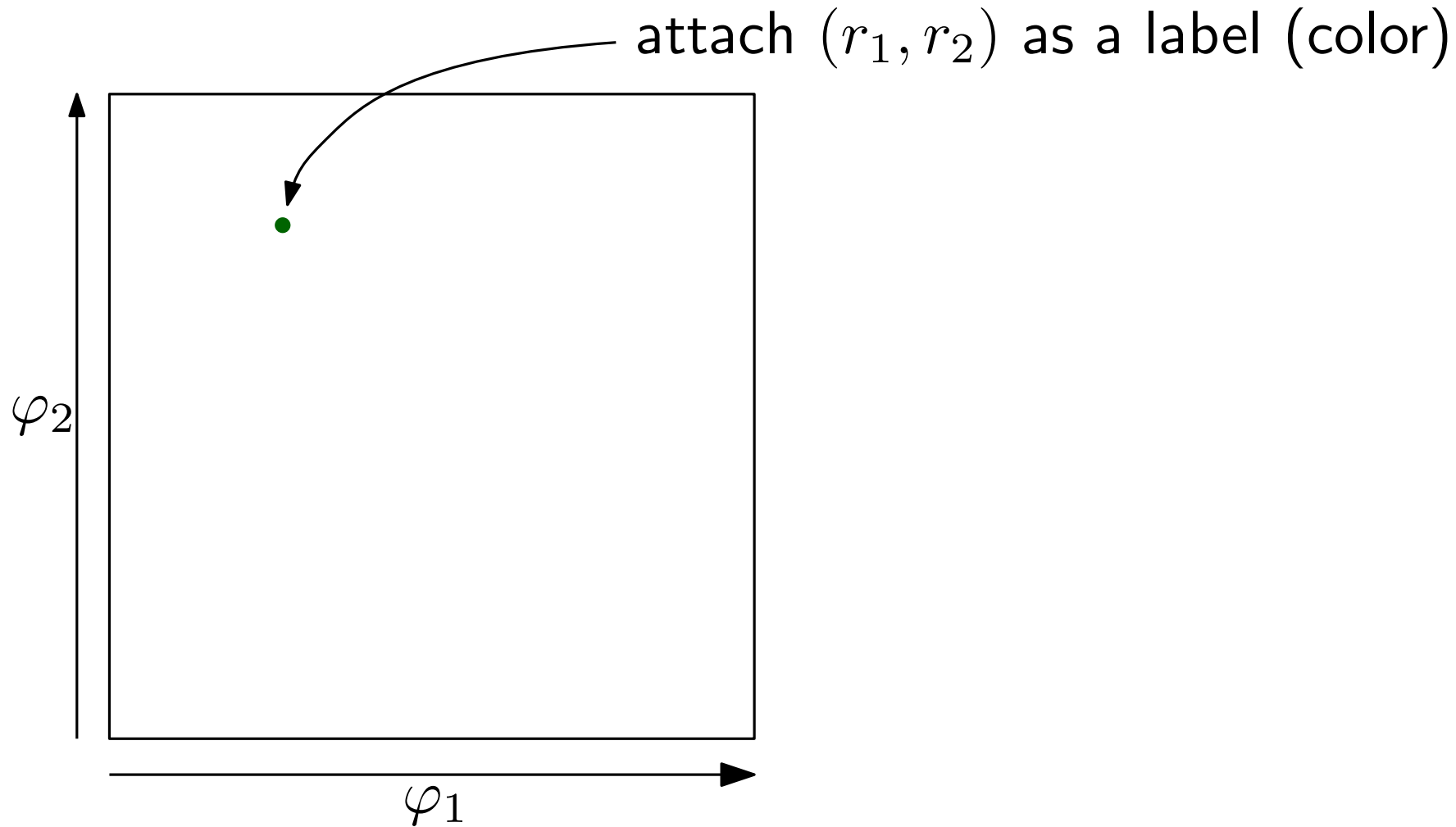
We have a plane  $P$  and we know its image in  $B$ .

“Double-polar” coordinates  $(r_1, \varphi_1, r_2, \varphi_2)$

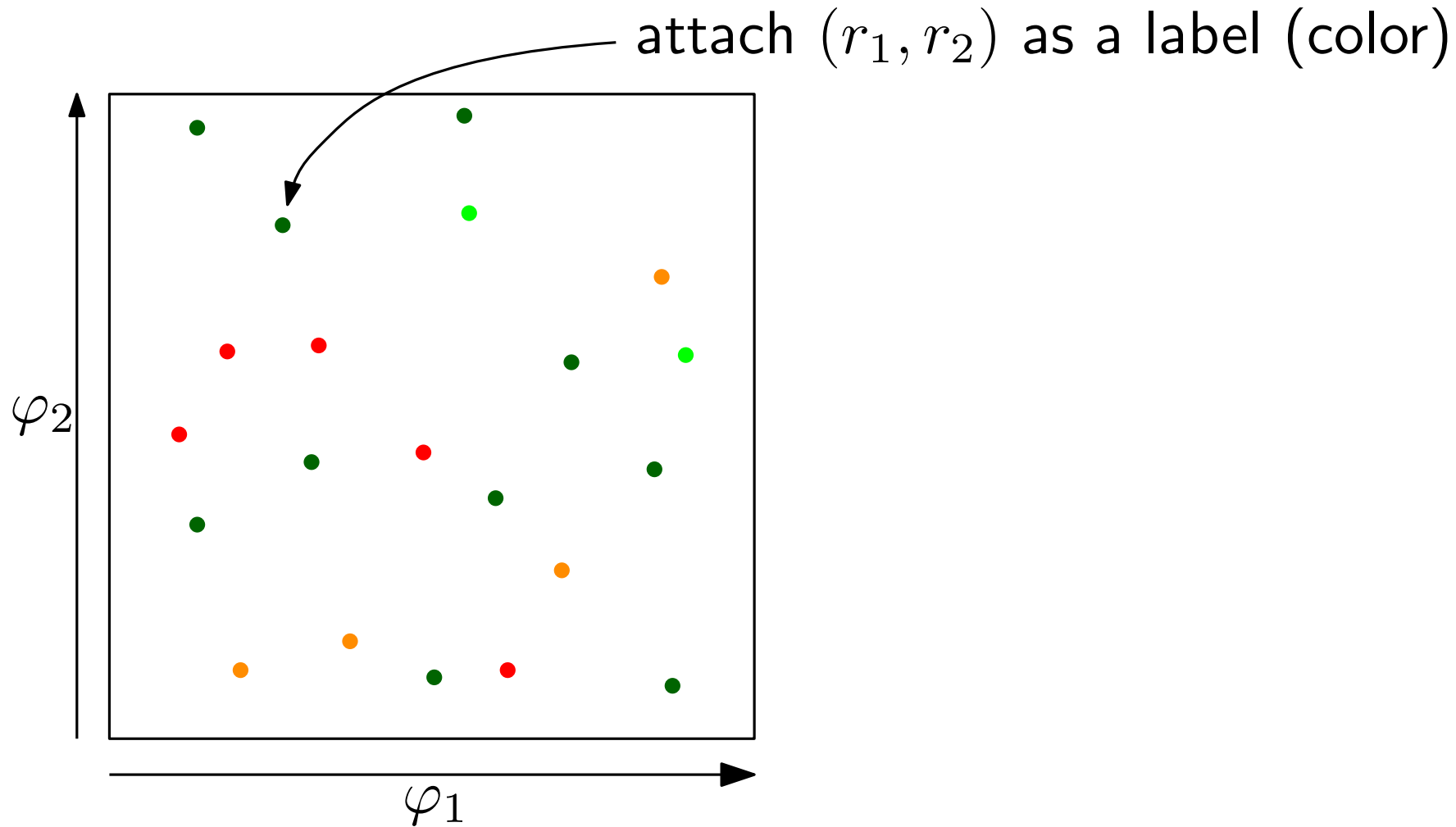
We can change  $\varphi_1$  and  $\varphi_2$ .  
 $r_1$  and  $r_2$  are fixed.



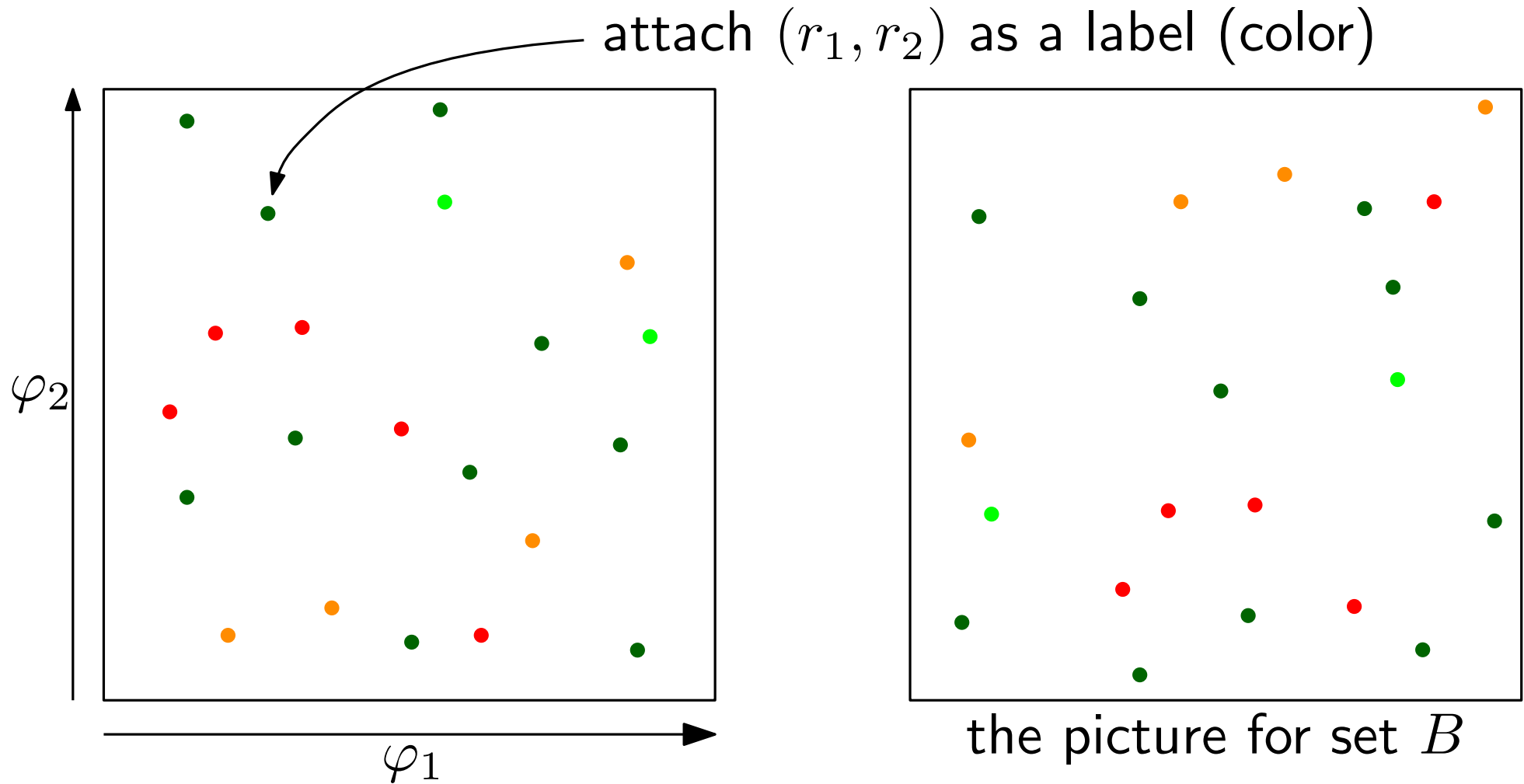
# 2+2 Dimension Reduction



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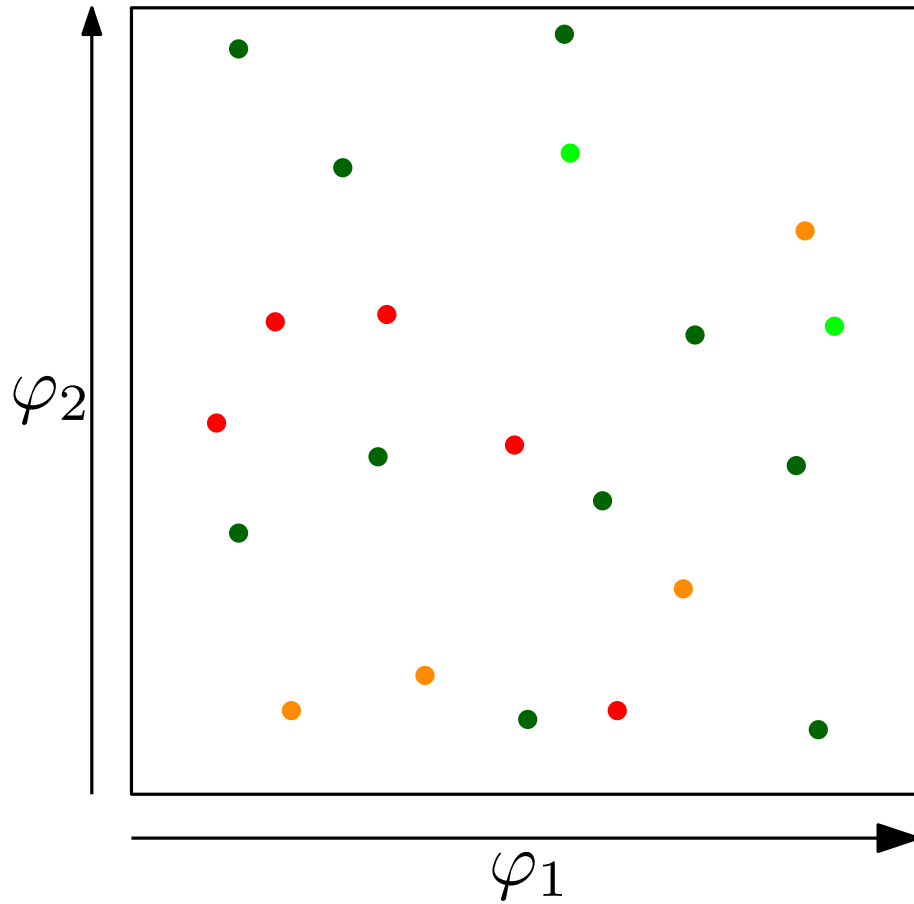


# 2+2 Dimension Reduction



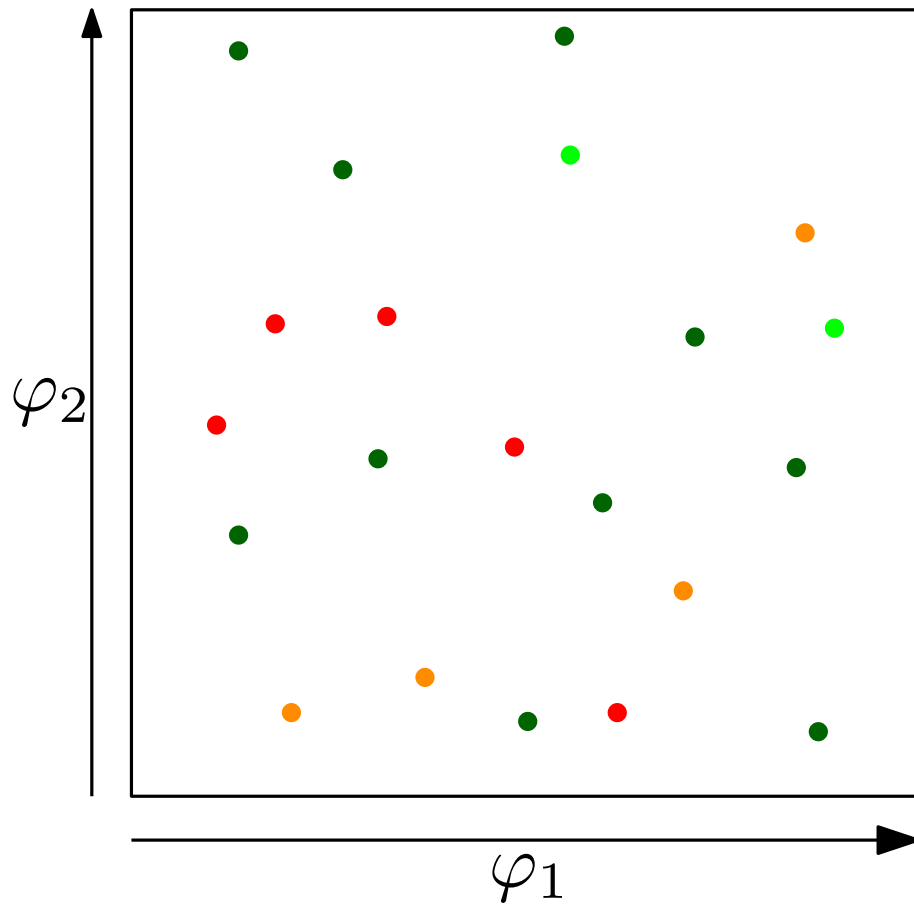
Are they the same up to translation on the  $\varphi_1, \varphi_2$ -torus?

Prune without losing information:  
(CANONICAL SET)



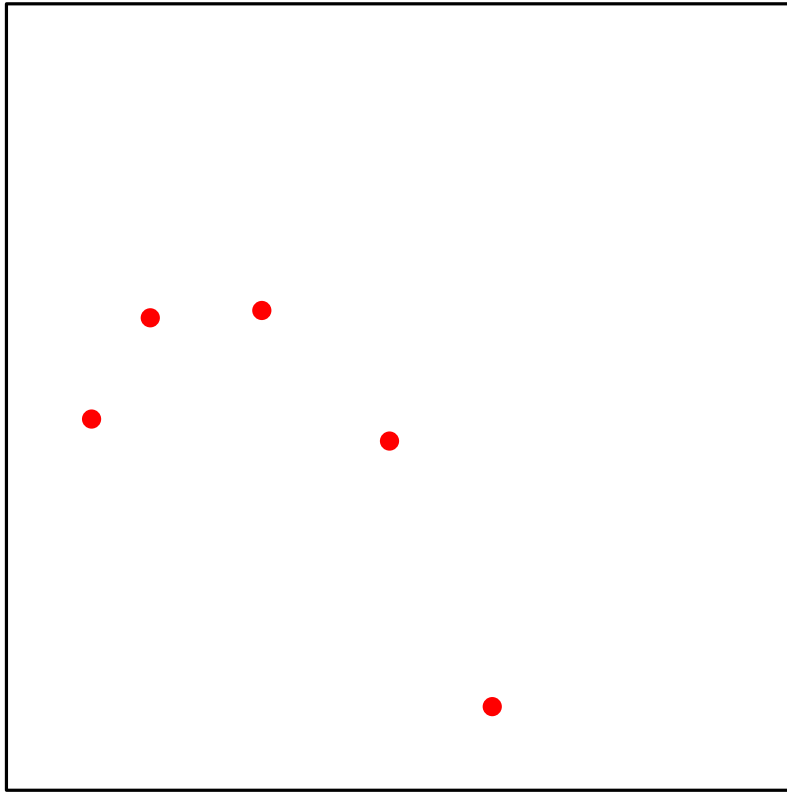
Prune without losing information:  
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Pick a color class



Prune without losing information:  
(CANONICAL SET)

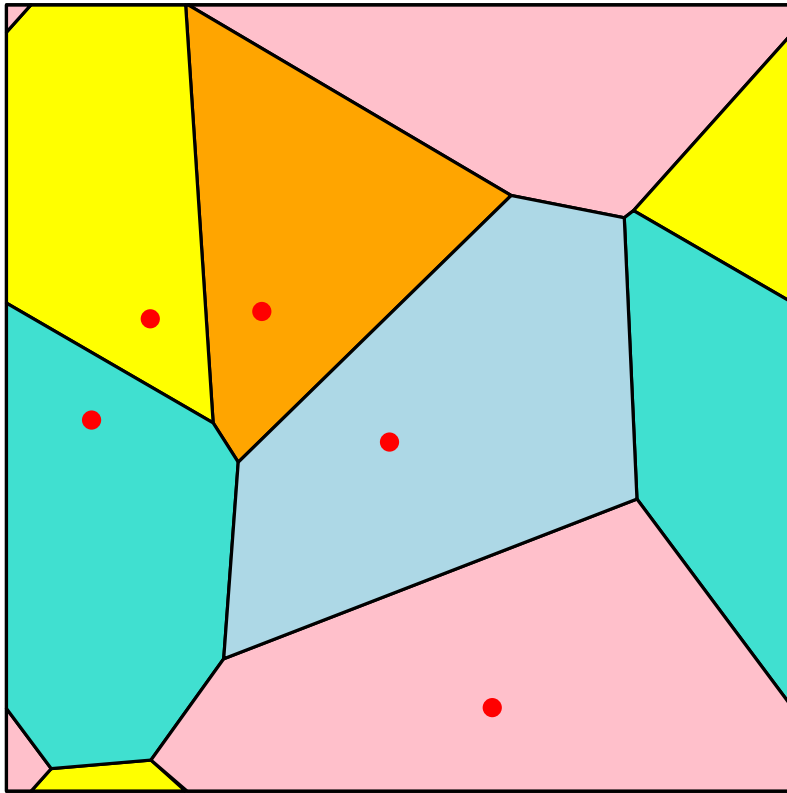
Pick a color class

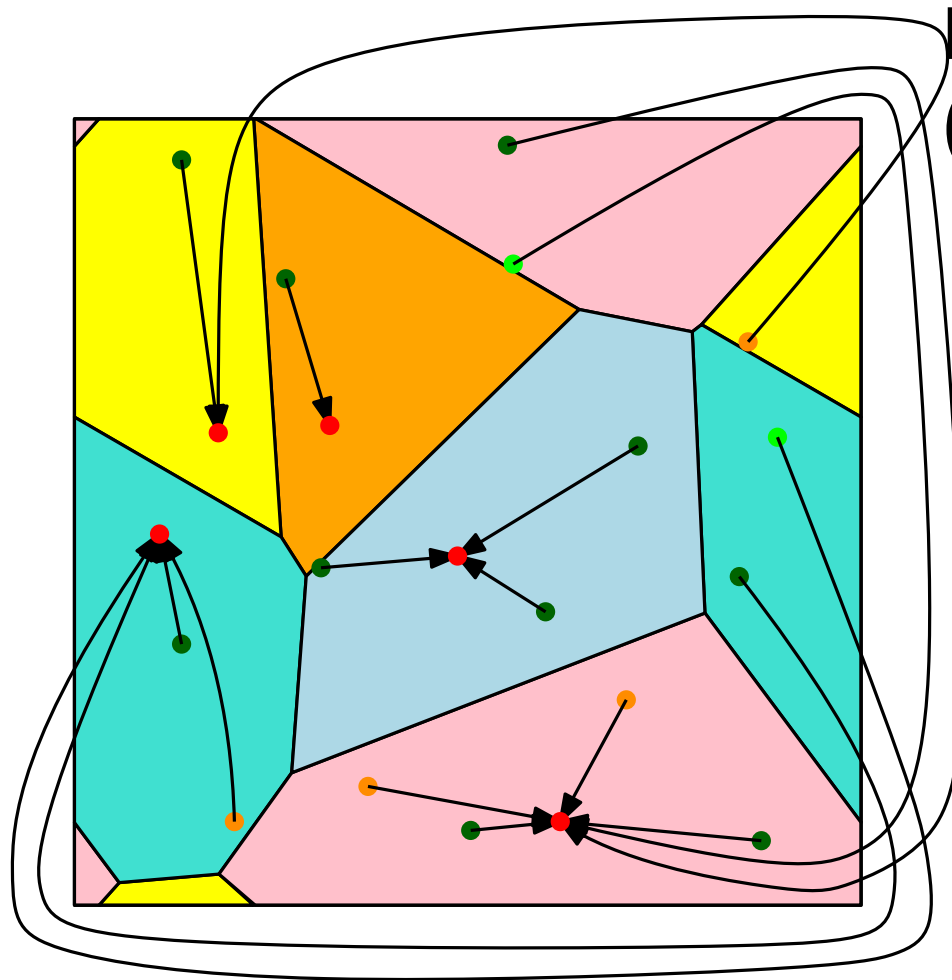


Prune without losing information:  
(CANONICAL SET)

Pick a color class

Compute the Voronoi diagram





Prune without losing information:  
(CANONICAL SET)

Pick a color class

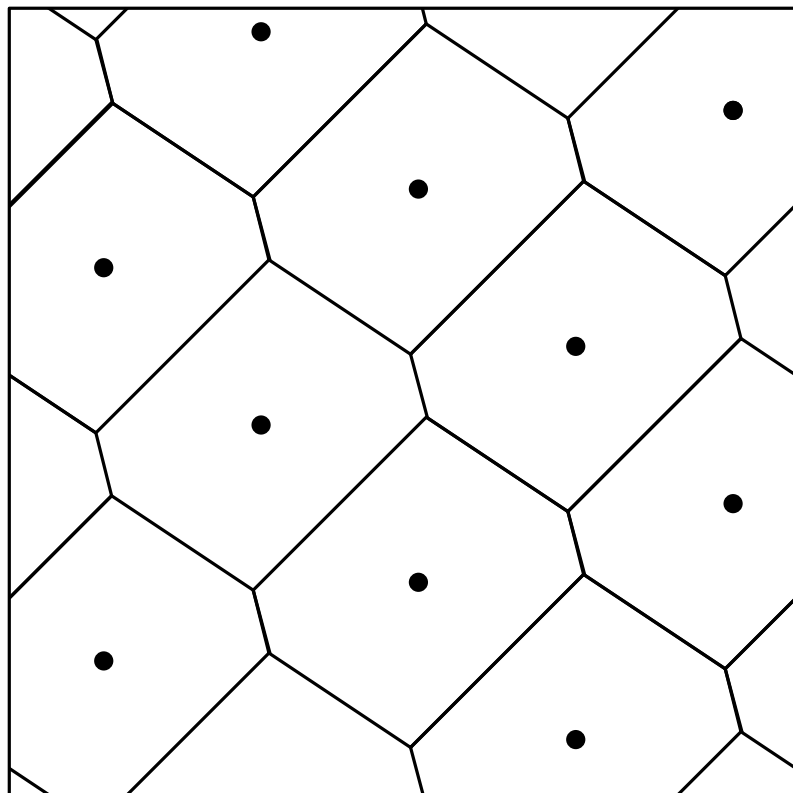
Compute the Voronoi diagram

Assign other points to cells.

Refine the coloring, based on  
color and relative position of  
assigned points, shape of  
Voronoi cell.

Repeat.

After recoloring, the reduced set has THE SAME translational symmetries as the old set.



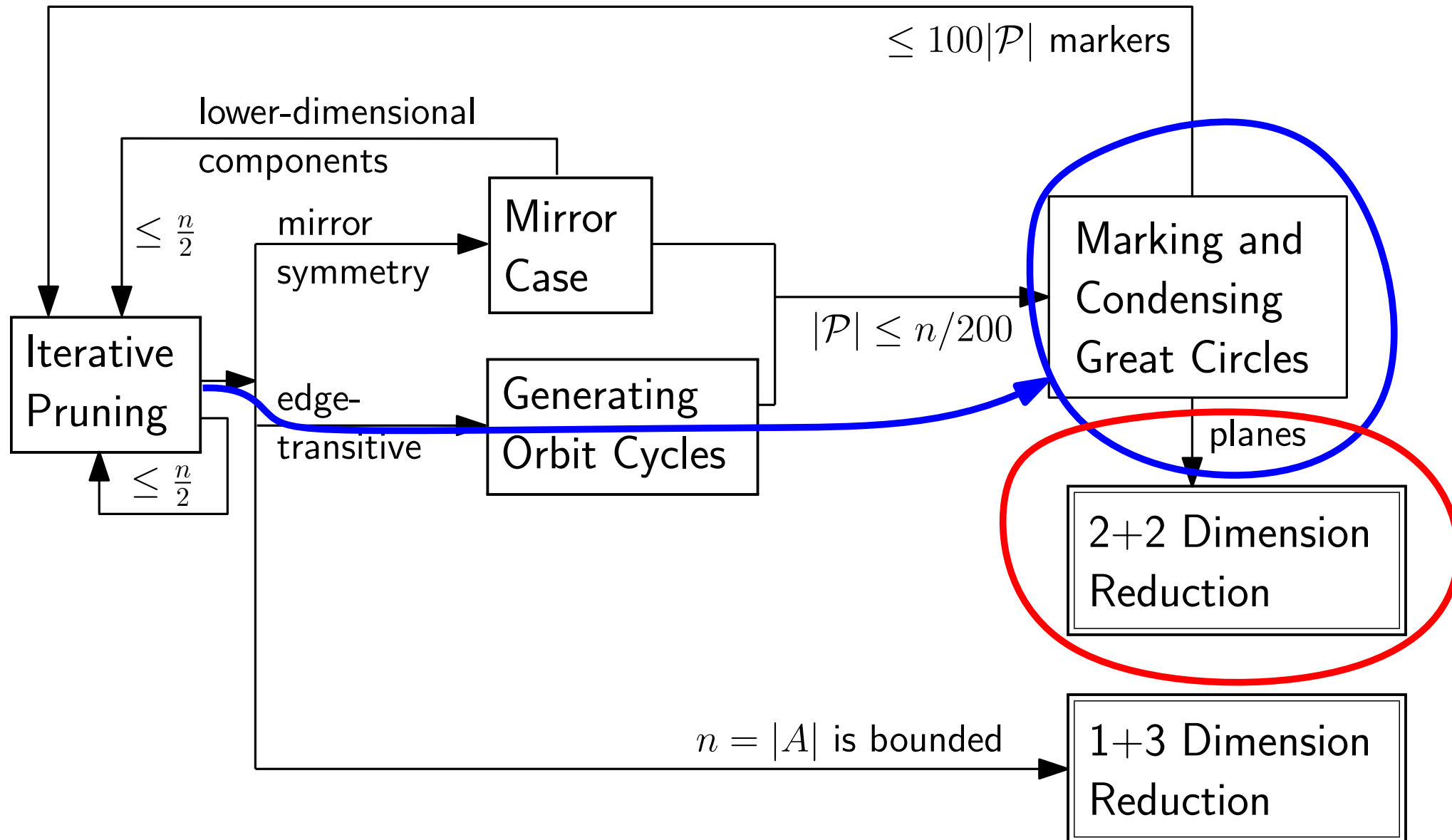
Termination:

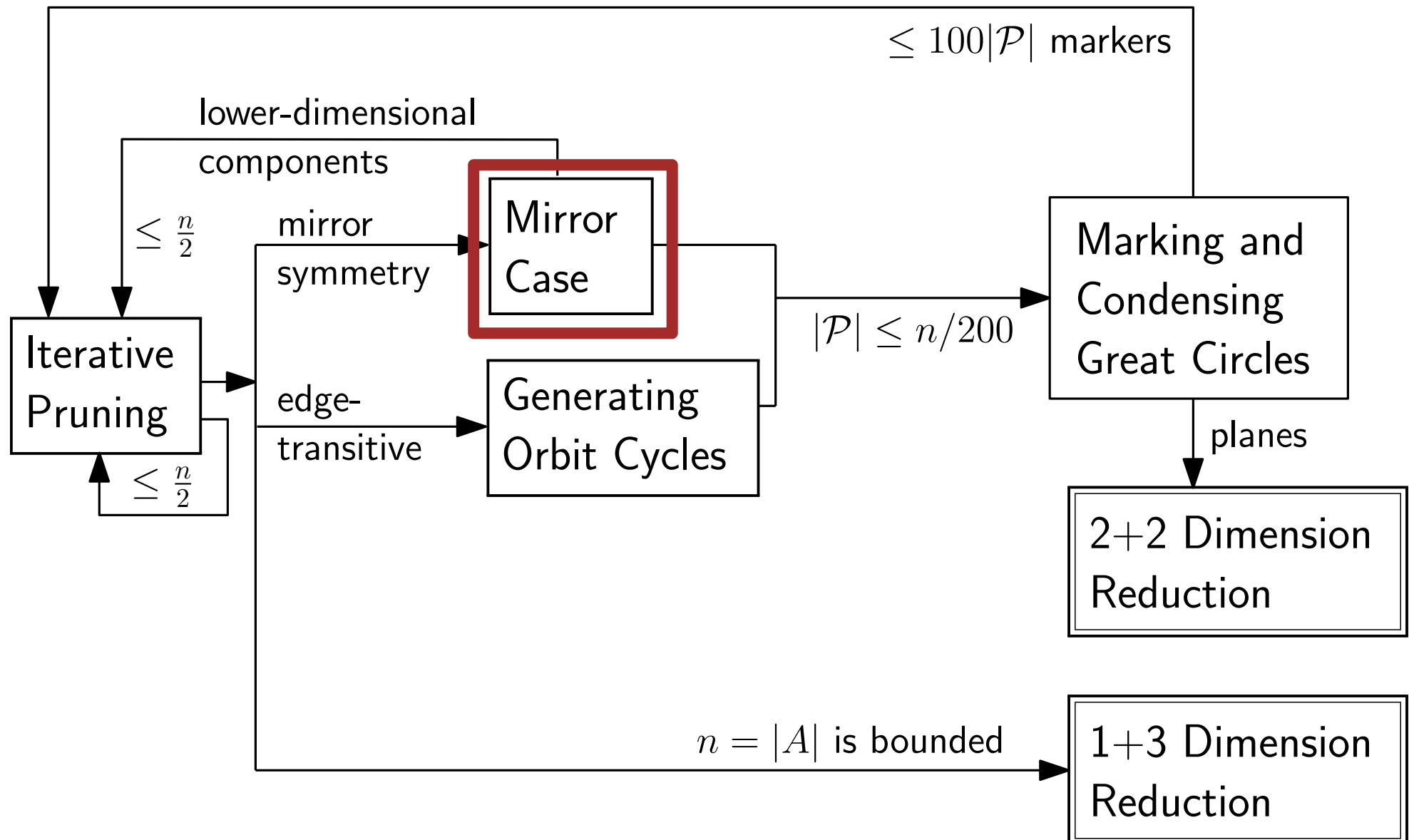
All points have the same color  
and the same cell shape  
(a modular *lattice*)

ANY point is as good a  
representative as any other.

CANONICAL SET  $c(A)$ :  
move (any) representative point  
to  $(\varphi_1, \varphi_2) = (0, 0)$ , or to  
 $(x_1, 0, x_3, 0)$ .

$$\exists T \text{ with } TP = P \text{ and } TA = B \iff c(A) = c(B)$$





Every edge acts like a perfect mirror of the neighborhood.

→ Every connected component is the orbit of a point under a group generated by reflections.

These groups have been classified. (Coxeter groups)

- “small” components  
→ pruning
- Cartesian product of 2-dimensional groups (infinite family)  
→ 2+2 dimension reduction
- “large” components  
→  $|A| \leq n_0$