Species of structures and Cayley's Formula

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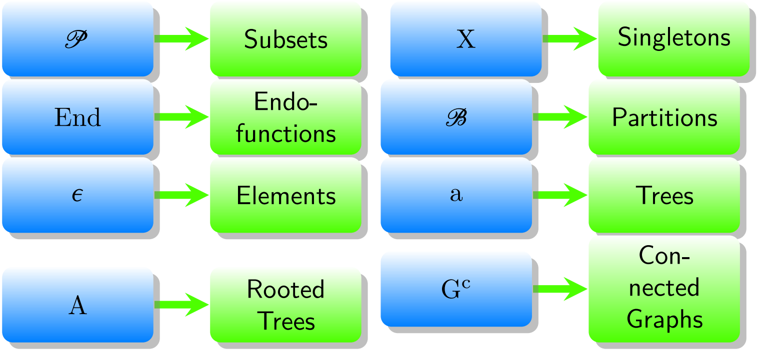
What is combinatorial Species

U	A finite set
α	A combinatorial structure: a permutation, a graph, a linear order, an endofunction, a set, a tree
Species	A functor(rule) F that takes an n element set U and produces the set $\mathrm{F[U]}$ of structures α of type F.
F	F transforms any bijection $\sigma : \mathrm{U} \mapsto \mathrm{V}$, to a bijection $\mathrm{F}[\sigma]$ (*transport*) between sets $\mathrm{F[U]}$ and $\mathrm{F[V]}$.

- S — Species of Permutations
- L — Species of Linear Orders
- G — Species of Graphs
- C — Species of Cycles
- E — Species of Sets


$\mathscr{P}$
Subsets
X
Singletons
End
Endo-functions
$\mathscr{B}$
Partitions
ϵ
Elements
a
Trees
A
Rooted Trees
G^c
Con-nected Graphs

An immediate consequence of existence of transport is that the cardinality of $F[U]$ for a finite set U depends only on $|U|$. In other words σ acts as a relabeling function on $F[U]$. For a species F, let $f_n = |F[\{1, \ldots, n\}]|$.

S	$f_n = n!$
L	$f_n = n!$
C	$f_n = (n-1)!$
E	$f_n = 1$

G

$$f_n = 2^{\binom{n}{2}}$$

X

$$f_n = 1 \text{ if } n = 1, 0 \text{ o.w.}$$

$\mathscr{P}$

$$f_n = 2^n$$

End

$$f_n = n^n$$

The ordinary generating function for a sequence $\{a_n\}$ is the formal power series

$$G(x) = \sum_{n \geq 0} a_n x^n$$

The exponential generating function for a sequence $\{a_n\}$ is the formal power series

$$F(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$$

$a_n = 1$

$$G(x) = \sum_{n \geq 0} x^n = \frac{1}{1-x}$$

$a_n = 1$

$$F(x) = \sum_{n \geq 0} \frac{x^n}{n!} = e^x$$

We usually associate three different generating series with a given species F.

- The exponential generating function $\mathrm{F(x)}$ for the labeled structures in $\mathrm{F[U]}$.
- The ordinary generating function $\bar{\mathrm{F}}(\mathrm{x})$ for the unlabeled structures (or isomorphism classes) in $\mathrm{F[U]}$
- The cycle index series, $\mathrm{Z_F(x)}$ for the sum of cycle indices of the automorphism groups of unlabeled structures in $\mathrm{F[U]}$.

For the purpose of this talk we will look only at the first.

Permutations	$S(x)$	$\sum n! \frac{x^n}{n!} = \frac{1}{1-x}$
Singletons	$X(x)$	x
Set	$E(x)$	e^x , $f_n = 1$
Linear Orders	$L(x)$	$\frac{1}{1-x}$
Cycles	C	$\log(\frac{1}{1-x})$, $f_n = n-1!$
Powerset	$\mathscr{P}(x)$	e^{2x}, $f_n = 2^n$
Endofunctions	$End(x)$	$\sum n^n \frac{x^n}{n!}$
Graphs	$G(x)$	$\sum 2^{\binom{n}{2}} \frac{x^n}{n!}$

Algebra of Species

Let F and G be two species. We aim to construct new species $\mathrm{F}+\mathrm{G}$, $\mathrm{F.G}$, $\mathrm{F}\circ\mathrm{G}$, and F' and F° so that their generating functions satisfy

- $(\mathrm{F}+\mathrm{G})(\mathrm{x}) = \mathrm{F}(\mathrm{x}) + \mathrm{G}(\mathrm{x})$
- $(\mathrm{F.G})(\mathrm{x}) = \mathrm{F}(\mathrm{x}).\mathrm{G}(\mathrm{x})$
- $(\mathrm{F}\circ\mathrm{G})(\mathrm{x}) = \mathrm{F}(\mathrm{G}(\mathrm{x}))$
- $\mathrm{F}'(\mathrm{x}) = \frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{F}(\mathrm{x}))$
- $\mathrm{F}^{\bullet}(\mathrm{x}) = \mathrm{x}\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{F}(\mathrm{x}))$

Algebra of Species

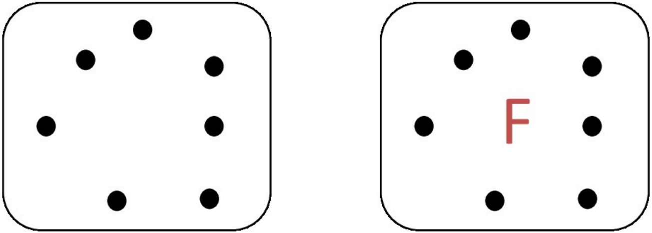


Figure: A set U and representation of $\mathrm{F}[\mathrm{U}]$

Sum of Species

We need to define species $(F + G)$ so that $(F + G)(x) = F(x) + G(x)$, i.e, the number of structures of $F+G$ of order n is $f_n + g_n$.

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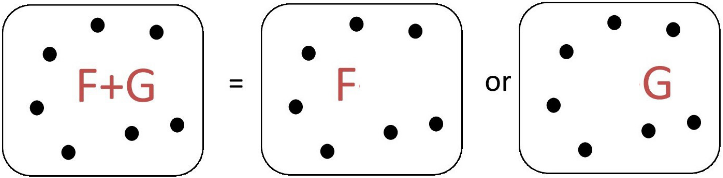

Figure: Representation of $F + G$

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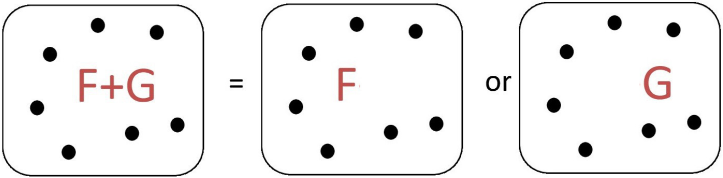

Figure: Representation of $F + G$

i.e., for finite set U, we take the disjoint union of $F[U]$ and $G[U]$.

E_o odd sets

$$E_o(x) = \sum_{n\geq 0} \frac{x^{2n+1}}{2n+1!} = \sinh(x)$$

E_e even sets

$$E_e(x) = \sum_{n\geq 0} \frac{x^{2n}}{2n!} = \cosh(x)$$

E Sets

$$E(x) = e^x = \cosh(x) + \sinh(x)$$

Product of Species

We need to define species $(F \cdot G)$ so that $(F \cdot G)(x) = F(x) \cdot G(x)$.

Product of Species

We need to define species $(\mathrm{F} \cdot \mathrm{G})$ so that $(\mathrm{F} \cdot \mathrm{G})(\mathrm{x}) = \mathrm{F}(\mathrm{x}) \cdot \mathrm{G}(\mathrm{x})$.

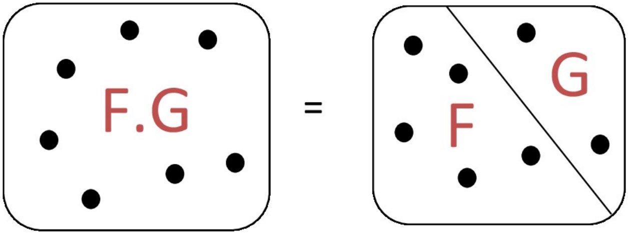

Figure: Representation of $\mathrm{F} \cdot \mathrm{G}$

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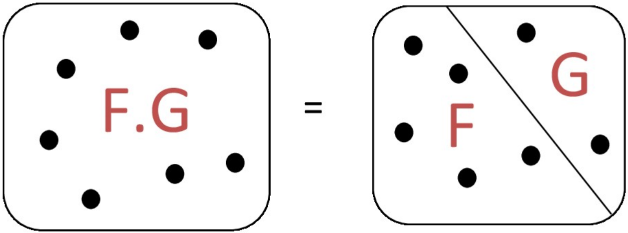


Figure: Representation of $F \cdot G$

i.e, the number of structures of $F \cdot G$ of order n is $\sum \binom{n}{i} f_i g_{n-i}$.

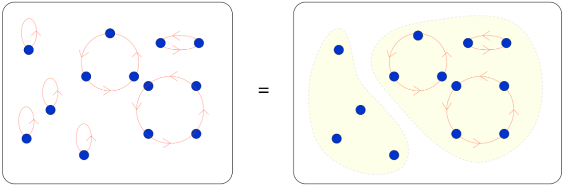
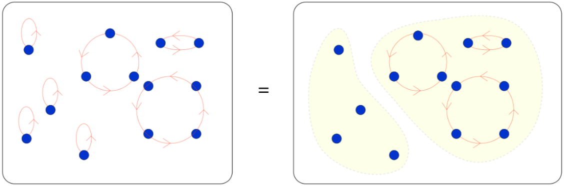


Figure: Proof that $\mathrm{S} = \mathrm{E} \cdot \mathrm{D}$



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D — Derangement: Permutations with no fixed point

$S(x)$ — $E(x) \cdot D(x)$

$D(x)$ — $e^{-x}/(1-x)$, $d_n = n! \sum_0^n \frac{(-1)^i}{i!}$

More examples to try

Let $\mathrm{F_k}$ denote the species F restricted to sets on cardinality k. Verify that

- $\mathrm{F} = \mathrm{F_0} + \mathrm{F_1} + \ldots, \mathrm{F_k}, \ldots,$
- $\mathscr{P} = \mathrm{E} \cdot \mathrm{E}$.
- $\mathscr{P}_\mathrm{k} = \mathrm{E_k} \cdot \mathrm{E}$

Composition of Species

If $G(0) = 0$, we define species $(F \circ G)$ so that $(F \circ G)(x) = F(G(x))$.

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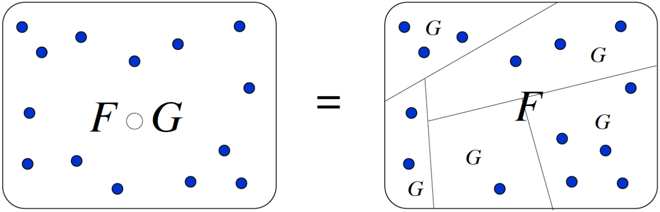

Figure: Representing $F \circ G$.

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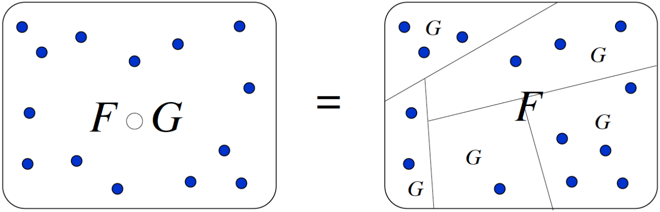

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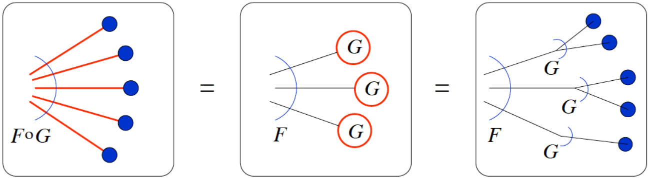

Figure: Another representation.

The number of structures of $F \circ G$ of order n is

$$\sum_{k=0}^{n} \sum_{n_1+n_2+..n_k=n} \frac{n!}{k!n_1!\ldots n_k!} f_k g_{n_1} g_{n_2} .. g_{n_k}$$

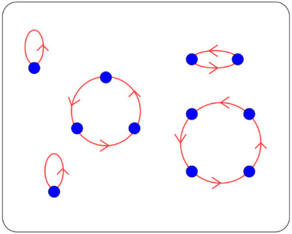

Figure: A permutation

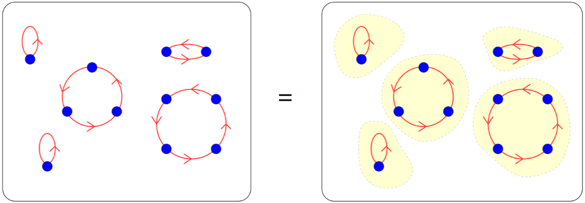

Figure: A permutation is a set of cycles

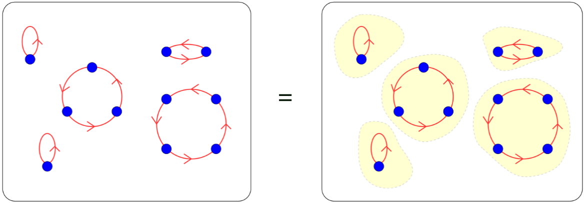

Figure: A permutation is a set of cycles

Thus,

$$S(x) = E(C(x)) = e^{C(x)} = e^{\log \frac{1}{1-x}} = \frac{1}{1-x}$$

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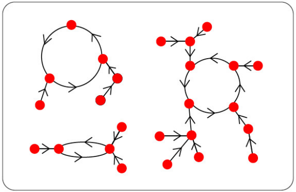

Figure: An endofunction

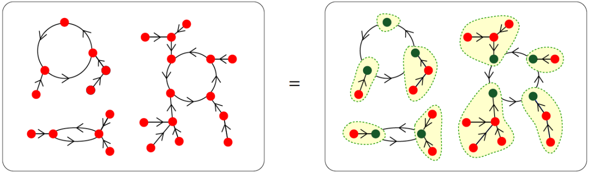

Figure: An endofunction is a permutation of rooted trees

Therefore, $\mathrm{End} = \mathrm{S} \circ \mathrm{A}$, where A is the species of rooted trees. And thus, $\mathrm{End}(\mathrm{x}) = 1/(1 - \mathrm{A}(\mathrm{x}))$.

- Find the generating function of $\mathscr{B}$, the species of partitions of a set, by writing it a composition of species.
- Find a relation between the generating functions of G and G^{c}.

Derivative

We need to define a species F' such that $F'(x) = \frac{d}{dx}F(x)$. Observe that the number of structures of F' of order n is f_{n+1}.

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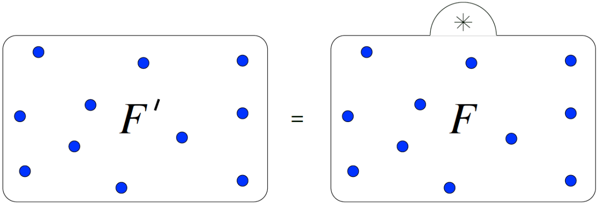

Figure: Derivative species

Derivative

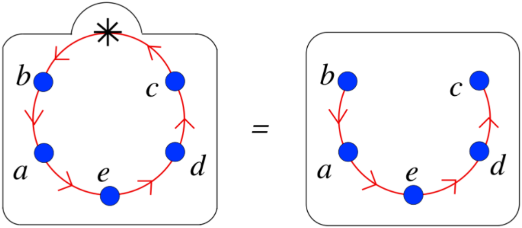

Figure: $\mathrm{C}' = \mathrm{L}$, Linear orders is derivative of cycle species

We already know $\mathrm{C}(\mathrm{x}) = \log \frac{1}{1-\mathrm{x}}$.

Now, $\mathrm{C}'(\mathrm{x}) = \frac{\mathrm{d}}{\mathrm{dx}} \log \frac{1}{1-\mathrm{x}} = \frac{1}{1-\mathrm{x}} = \mathrm{L}(\mathrm{x})$.

Verify that

- $\mathrm{E}' = \mathrm{E}$
- $\mathrm{L}' = \mathrm{L} \cdot \mathrm{L}$
- $(\mathrm{F} + \mathrm{G})' = \mathrm{F}' + \mathrm{G}'$
- $(\mathrm{F} \cdot \mathrm{G})' = \mathrm{F}' \cdot \mathrm{G} + \mathrm{F} \cdot \mathrm{G}'$
- $(\mathrm{F} \circ \mathrm{G})' = (\mathrm{F}' \circ \mathrm{G}) \cdot \mathrm{G}'$

Pointing

Pointing is the operation where we designate one of the n elements of U as special in $F[U]$. Thus $F^{\bullet}[U]$ will have cardinality nf_n.

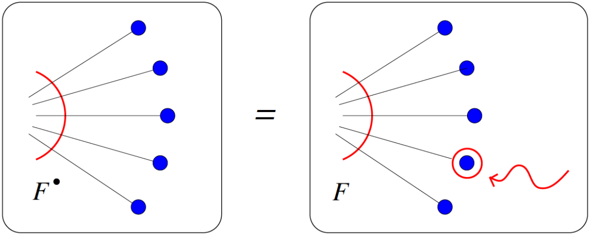

Figure: Representing $F^{\bullet}$

Pointing

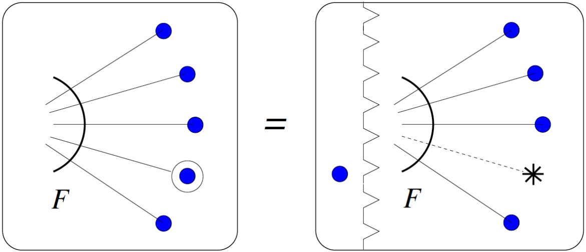

Figure: Representing $\mathrm{F}^{\bullet}$ in terms of derivative

Let $\mathbb{a}$ be the species of all trees. Let $\mathbb{A}$ be the species of all rooted trees. We can immediately see,

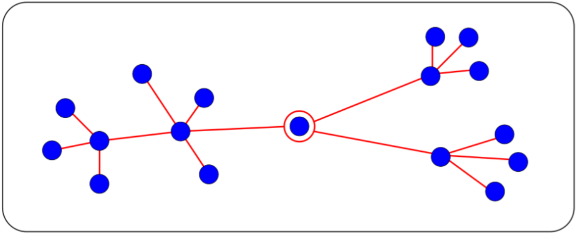

Figure: Species $\mathbb{A}$ of rooted trees is $\mathbb{a}^{\bullet}$

Cayley's formula

We now prove Cayley's formula that there are n^{n-2} labeled trees.

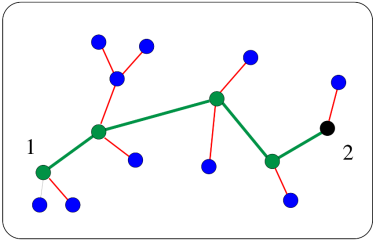

Figure: Species V of vertebrates is $a^{\bullet\bullet}$

That is $|V[n]| = n \cdot n \cdot |a[n]|$.

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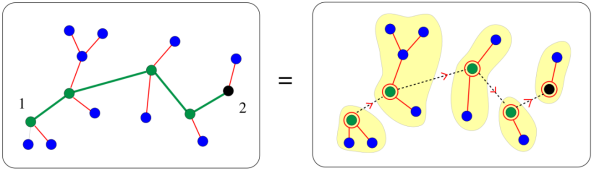
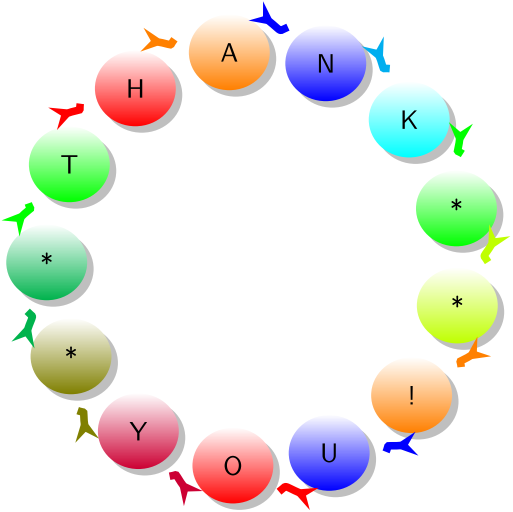


Figure: It may be seen as a linear order or rooted trees

That is, we have $V = L(A)$ and hence,
$V(x) = L(A(x)) = 1/(1 - A(x))$.
But we have already shown that $End(x) = 1/(1 - A(x))$. Thus,
$n^2 \cdot |a[n]| = |V[n]| = |End[n]| = n^n$. Therefore, $|a[n]| = n^{n-2}$.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References

1. Lectures of Xavier Viennot (now available at youtube matsciencechannel).
2. Book : Species of Combinatorial Structures by F. Bergerone, G. Labelle, and P. Leroux.