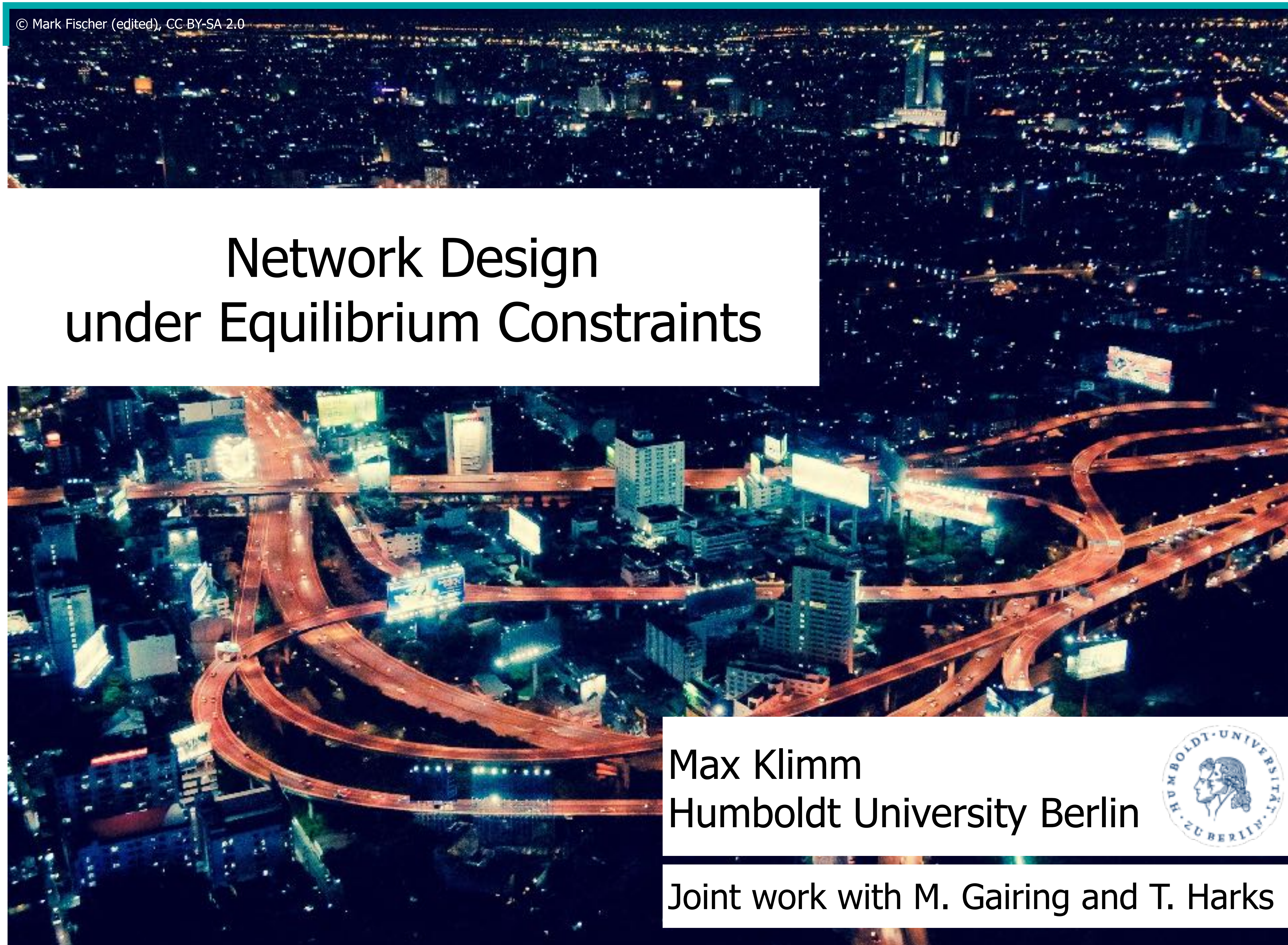


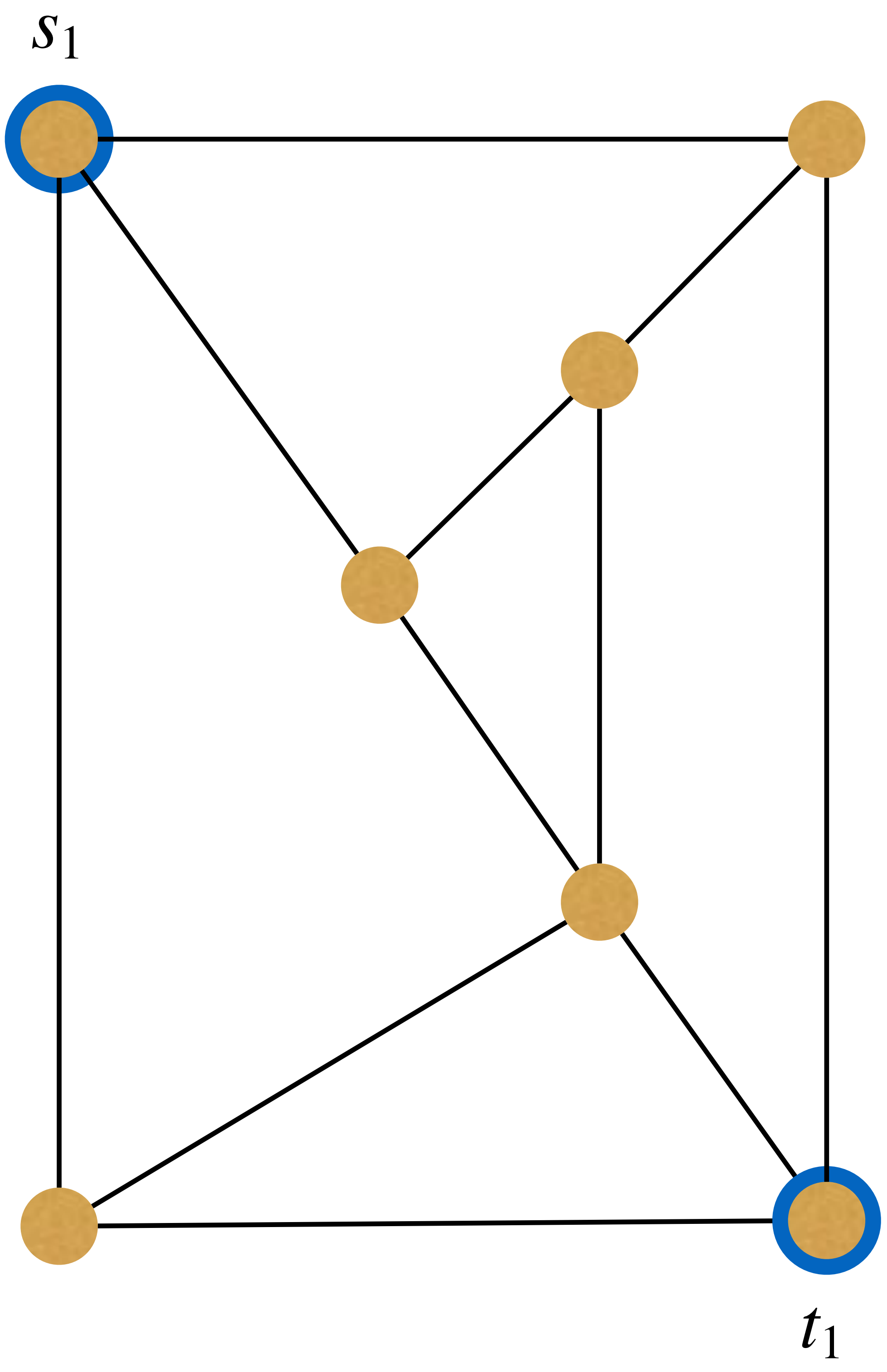
Network Design under Equilibrium Constraints

Max Klimm
Humboldt University Berlin

Joint work with M. Gairing and T. Harks

Selfish flows

- Graph $G = (V, E)$
  - edge cost functions $c_e : \mathbb{R}_+ \to \mathbb{R}_+$
    (strictly increasing, semi-convex, differentiable)
  - commodities $(s_i, t_i, d_i) \in \mathrm{V} \times \mathrm{V} \times \mathbb{R}_+$



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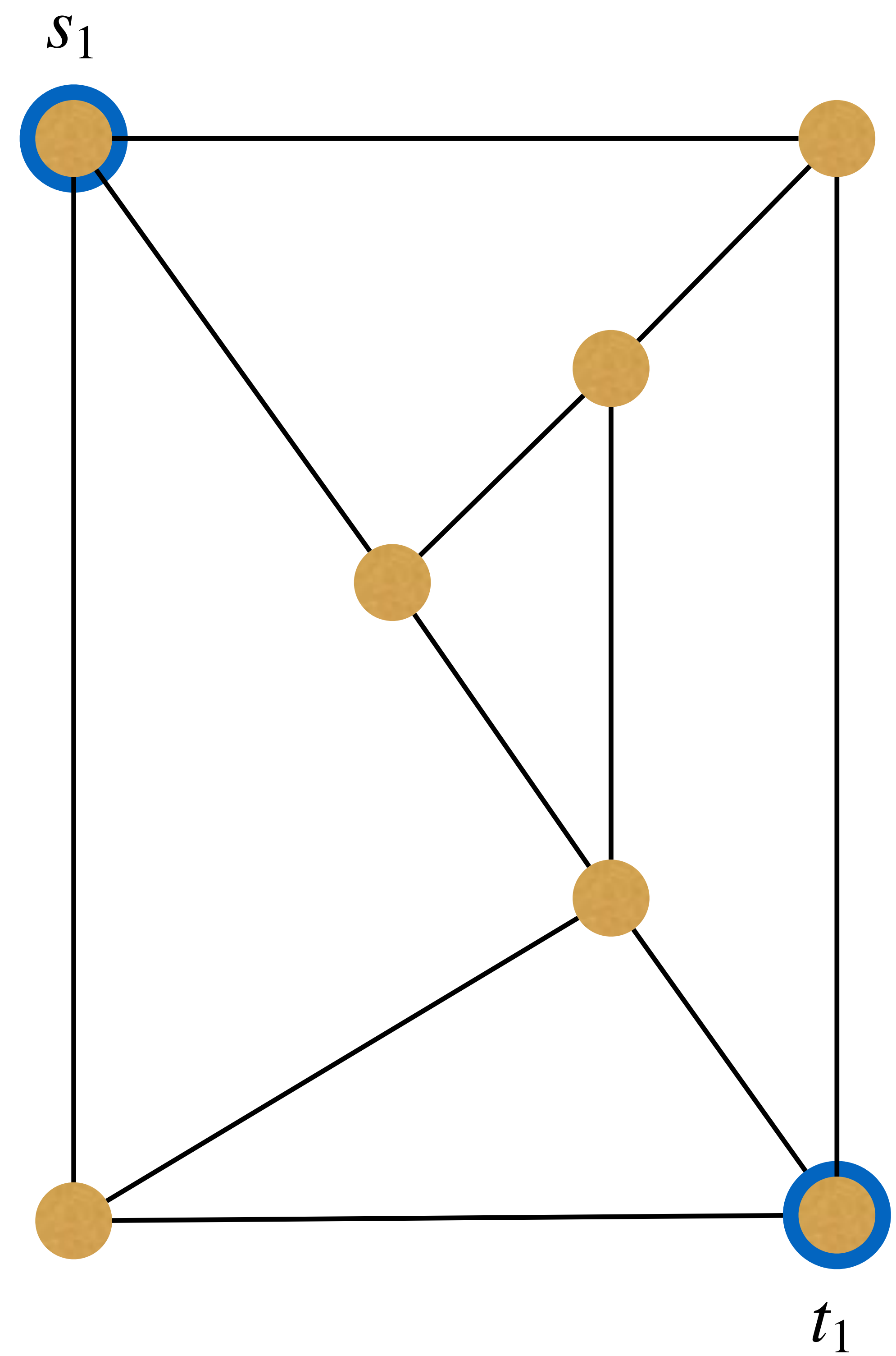

Definition — Wardrop equilibrium
Multicommodity flow $\mathbf{f} = (f_{i,P})$ with

$$\sum_{e \in P} c_e(f_e) \leq \sum_{e \in Q} c_e(f_e)$$

for all (s_i,t_i)-paths P,Q with $f_{i,P} > 0$.

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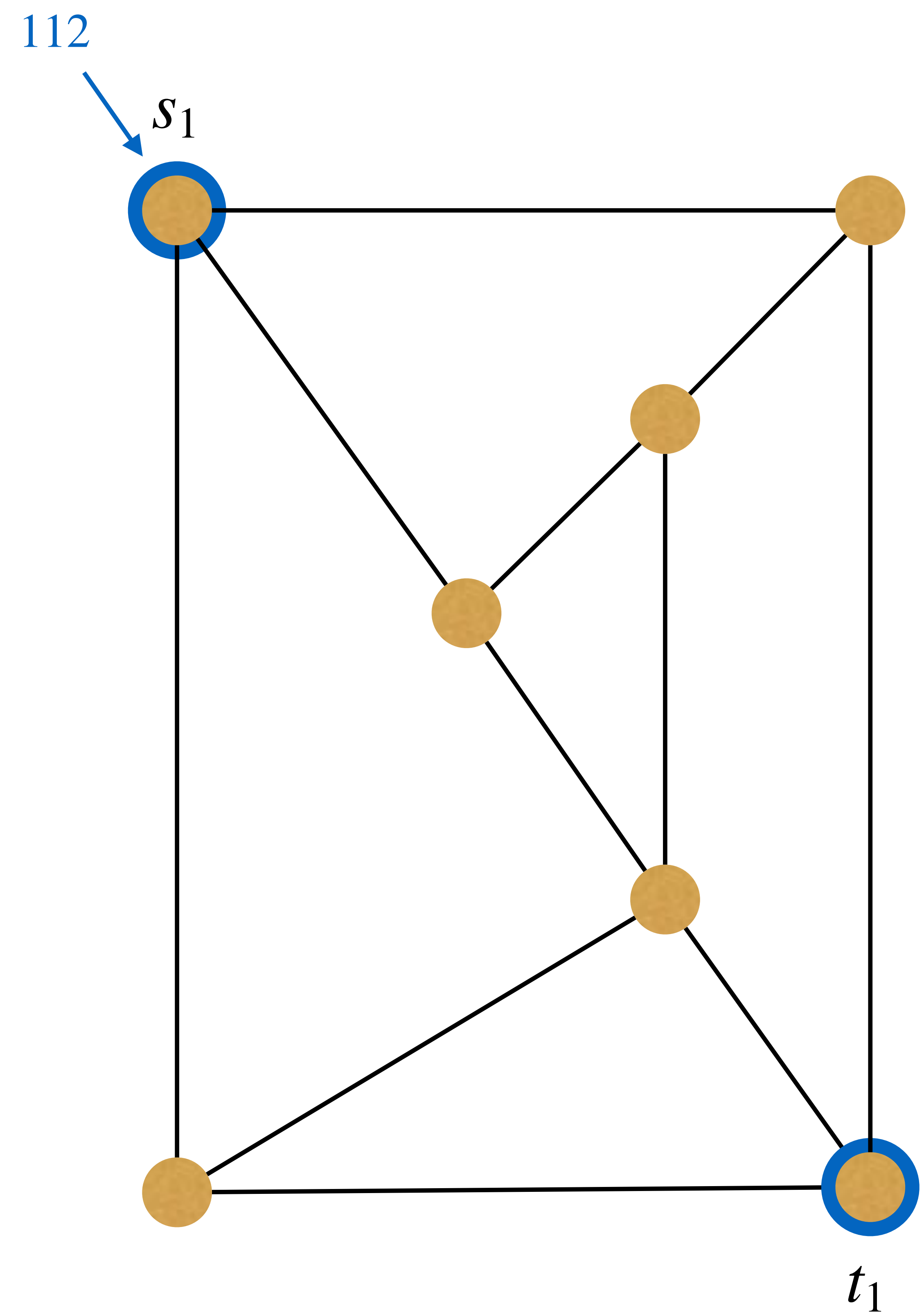

$c_e(x) = x$ for all e

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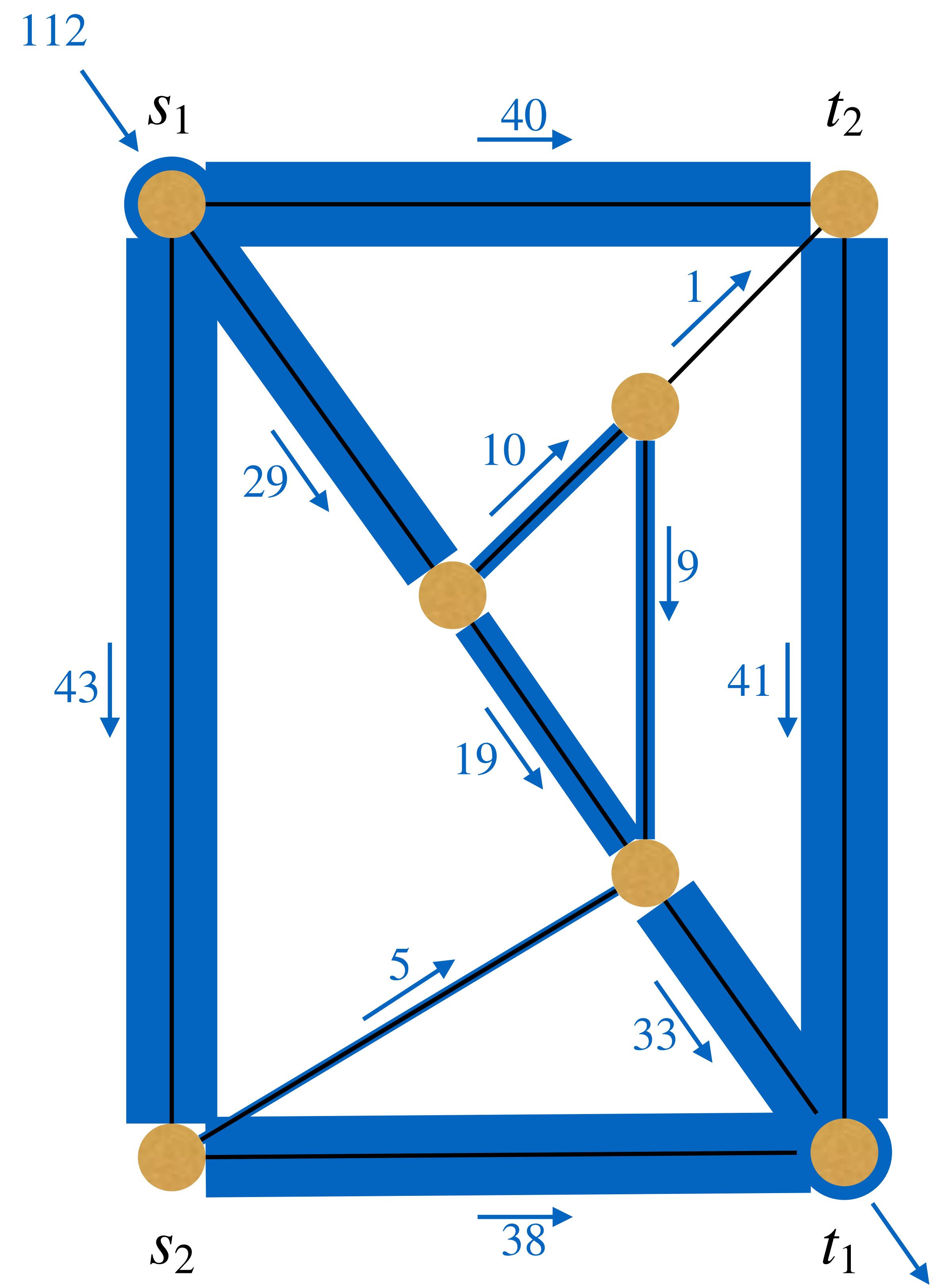
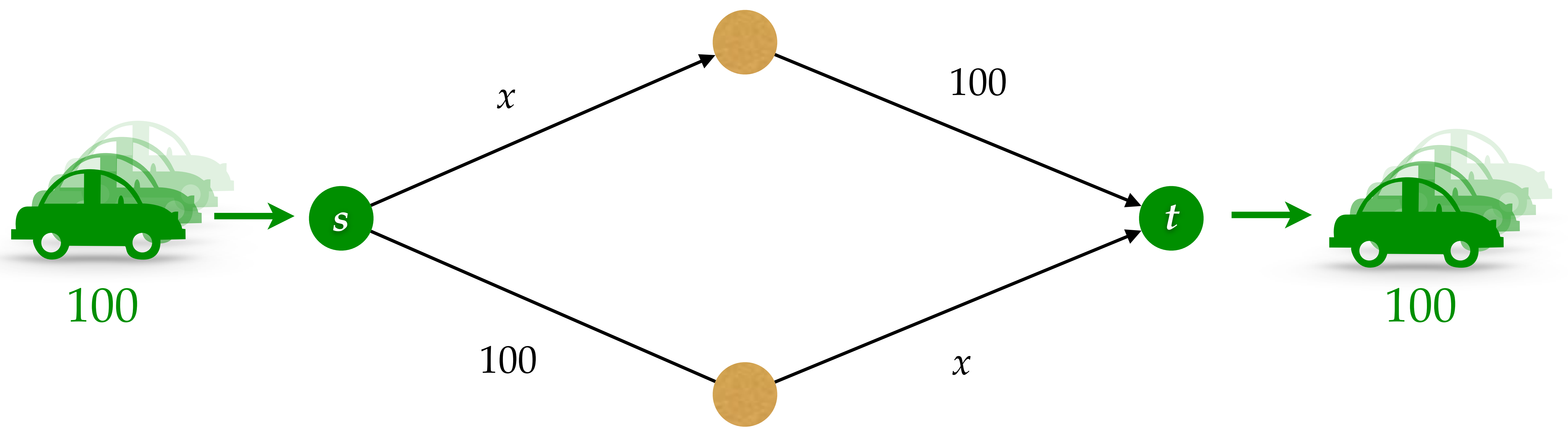


Braess' Paradox



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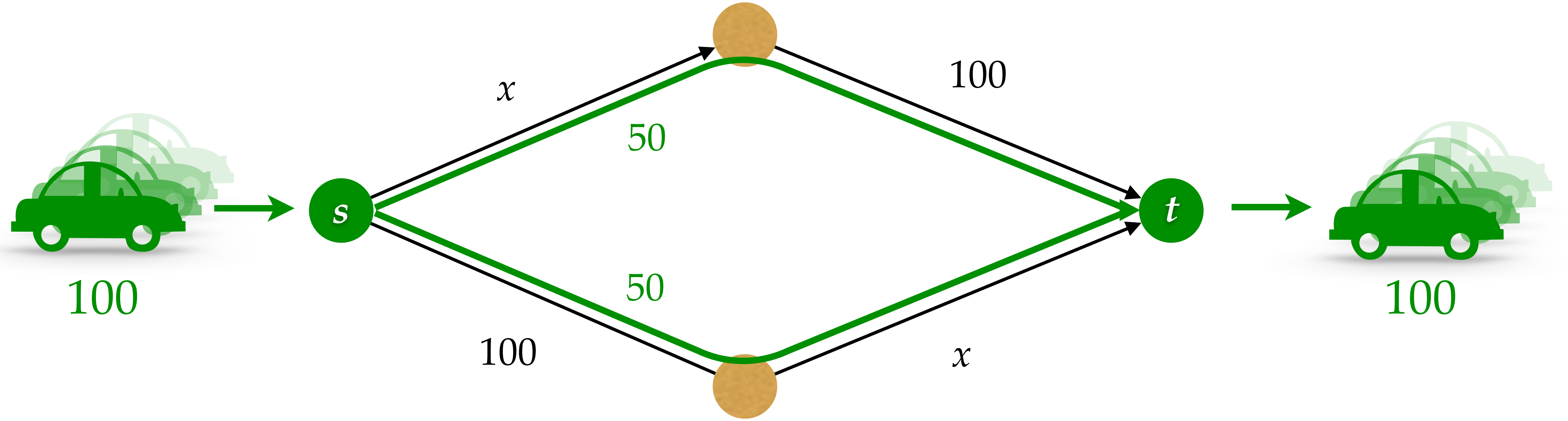

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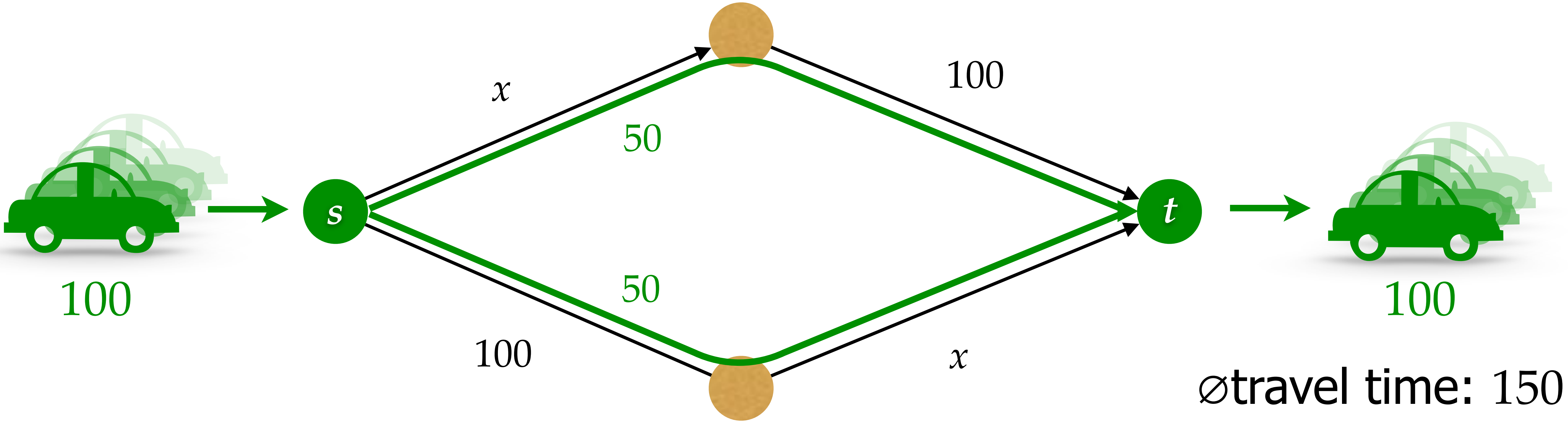

∅travel time: 150

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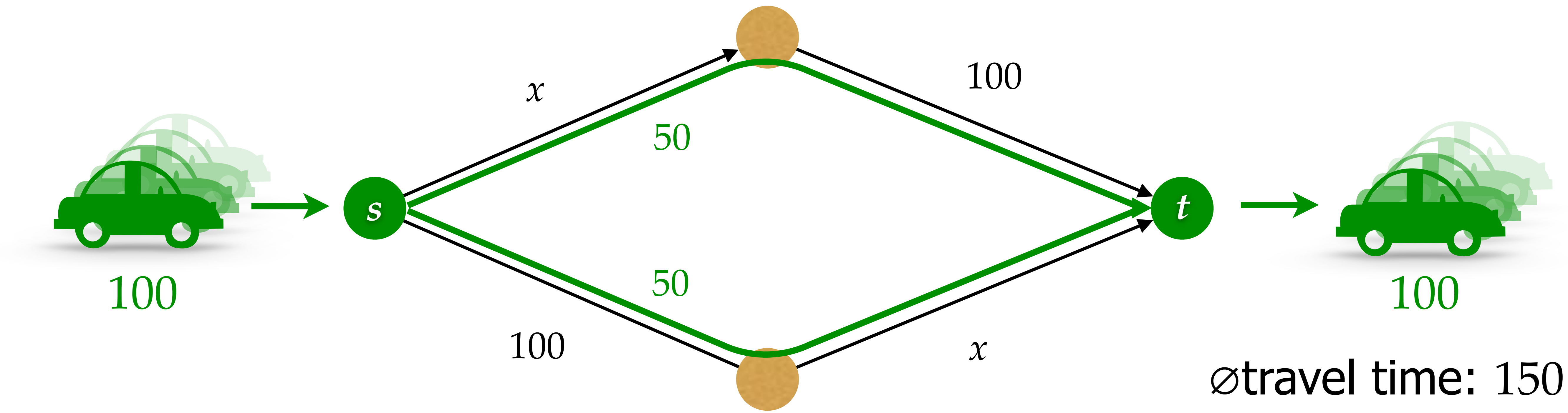
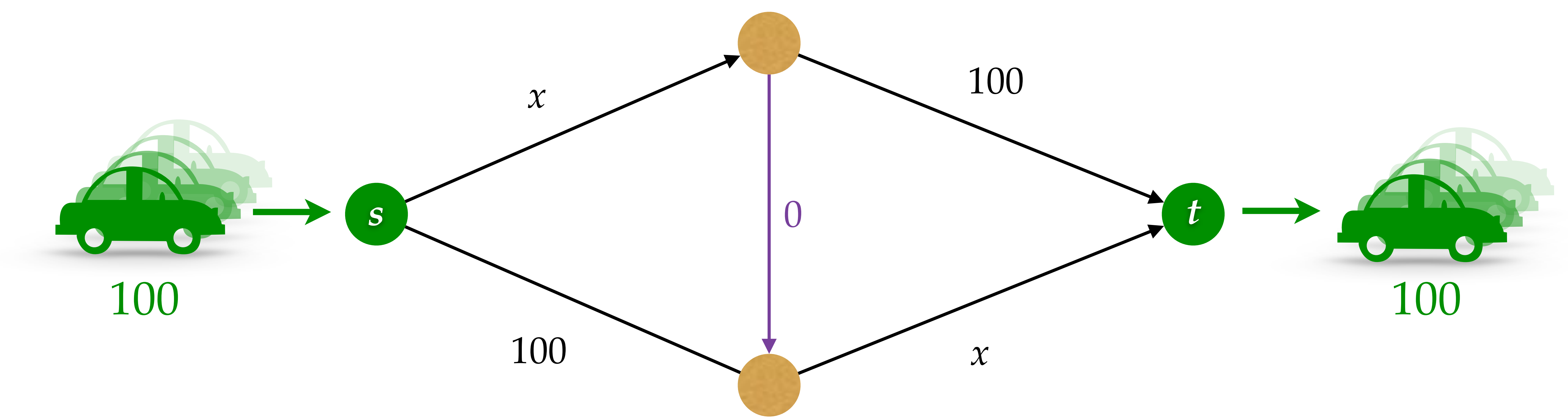


⌀travel time: 150



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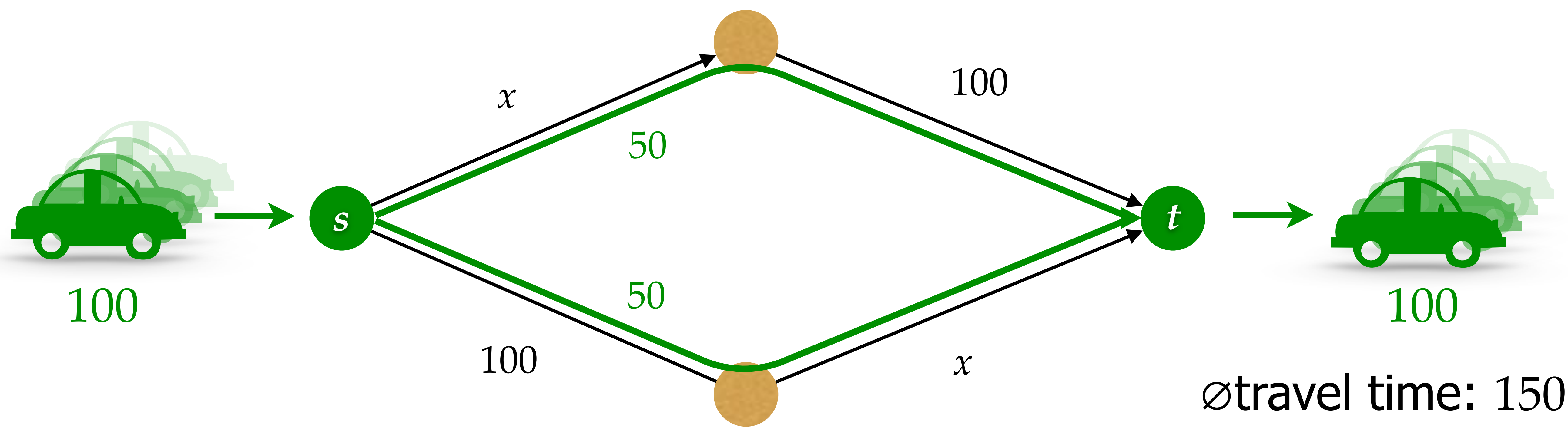

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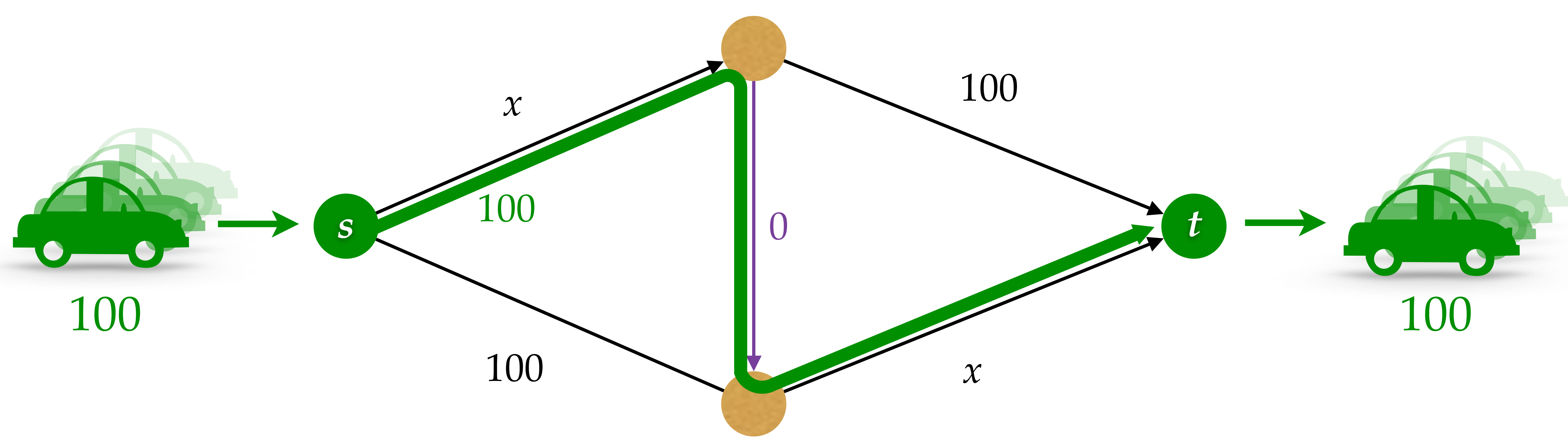

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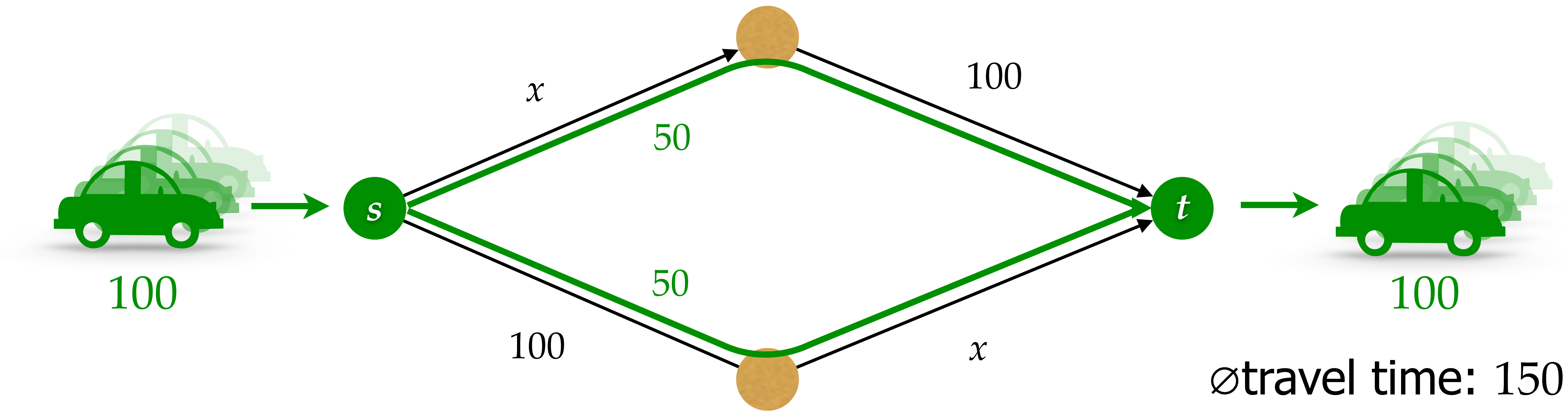

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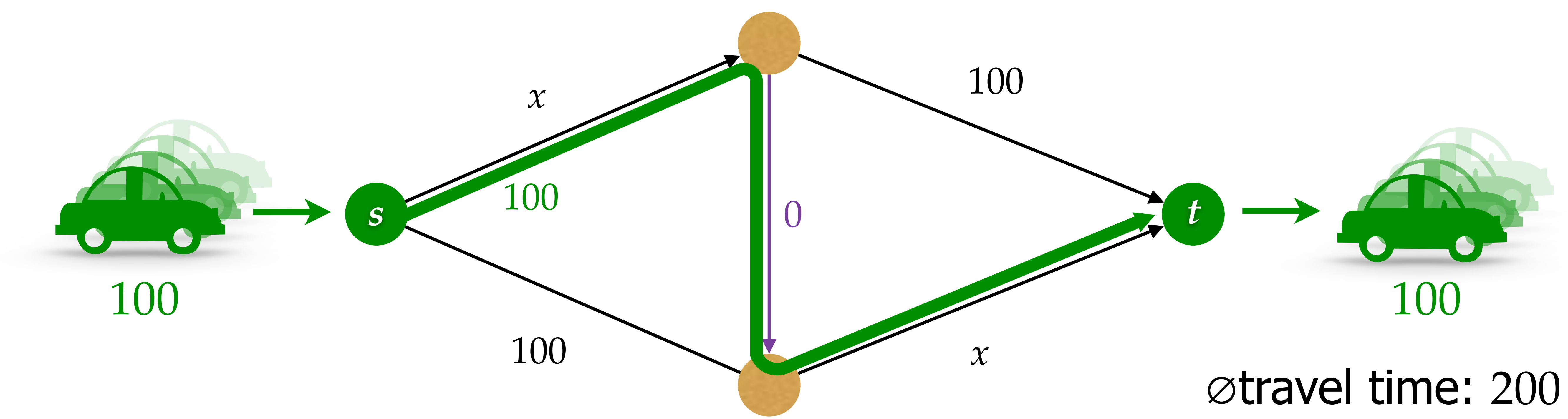

⌀travel time: 200

How can road networks be designed so that good equilibria emerge?

Continuous network design problem

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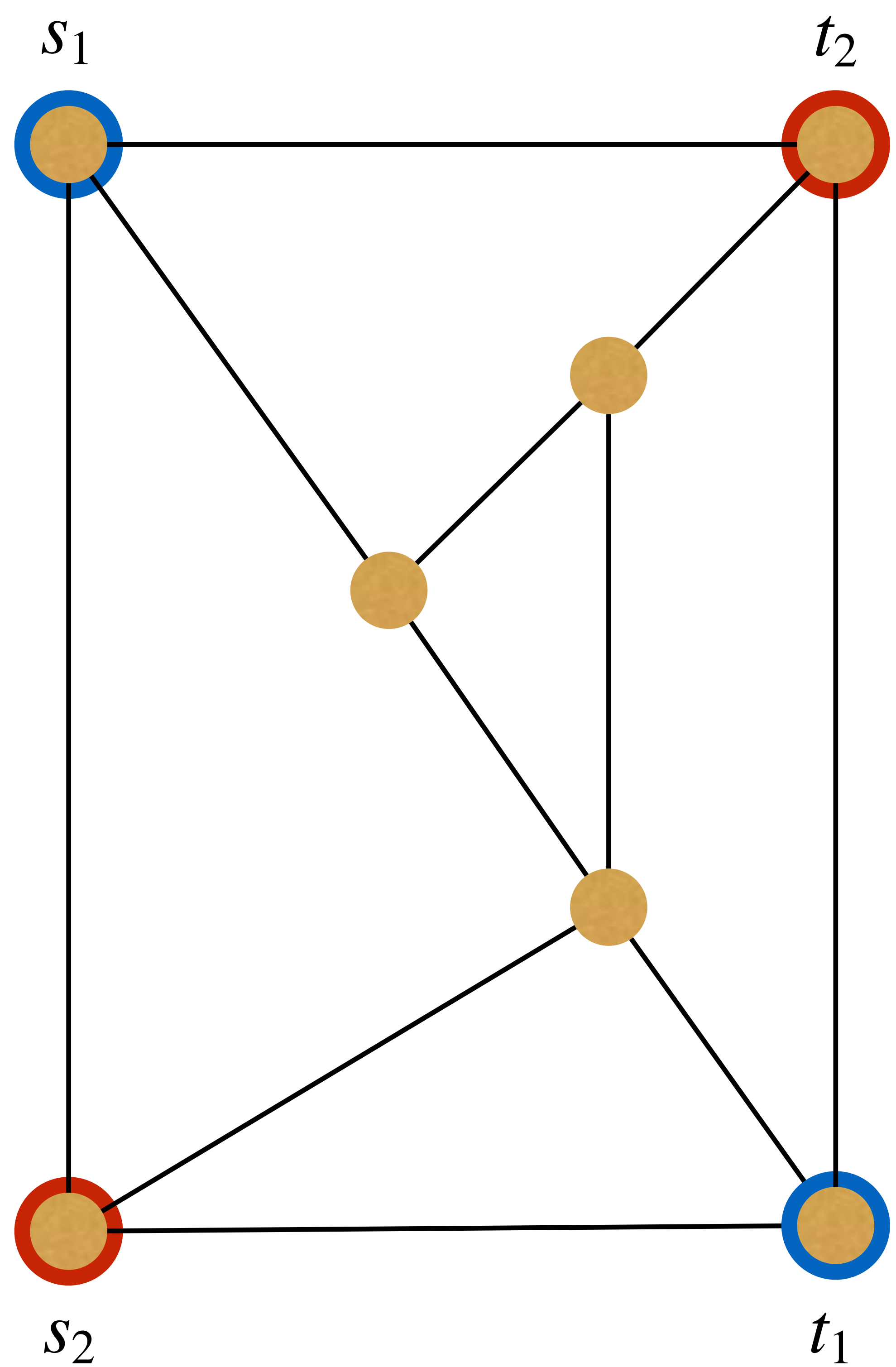

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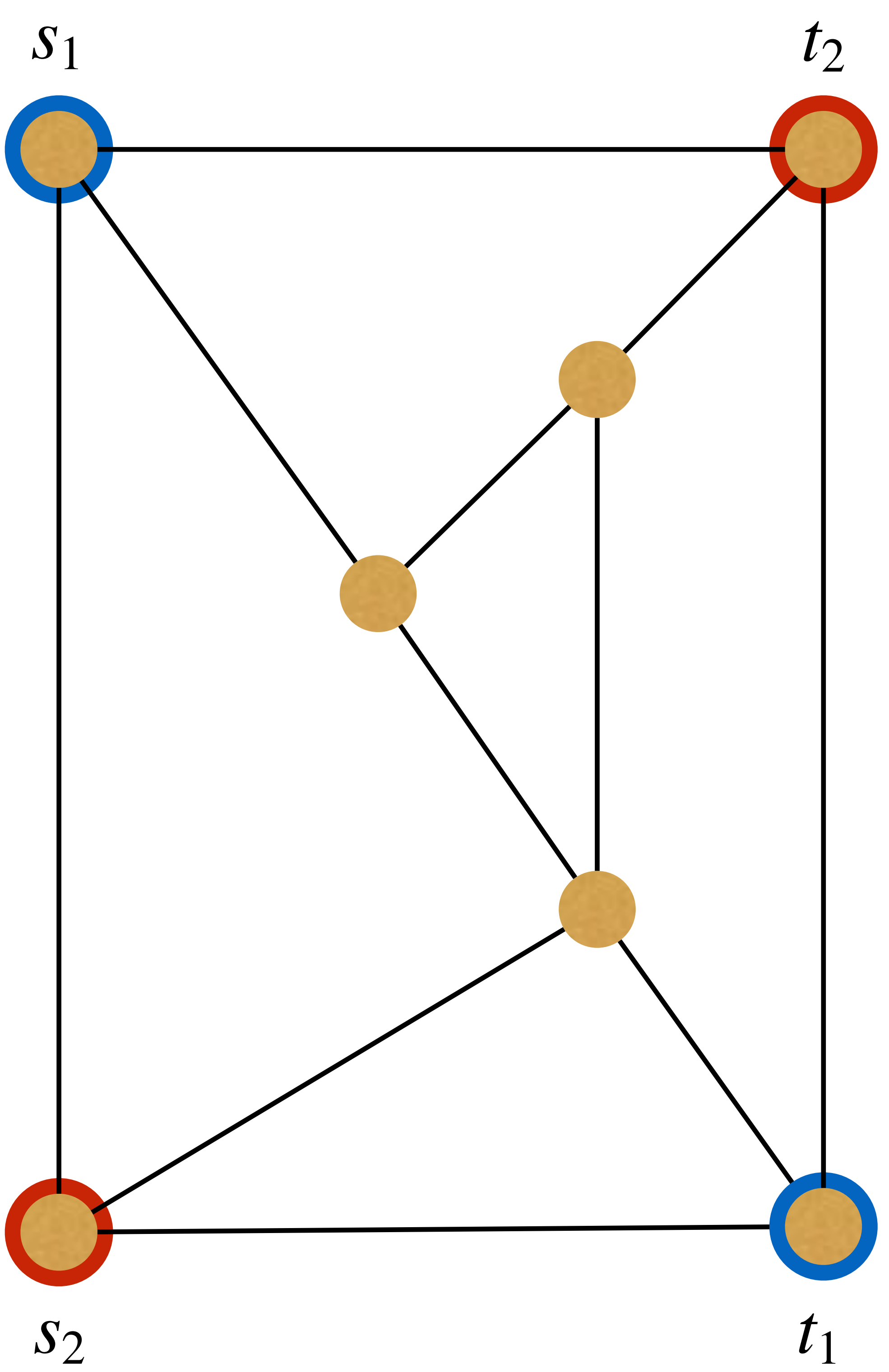
Continuous network design problem

$$\min_{\mathbf{f},\, \mathbf{z}} \sum_{e \in E} \left(c_e(f_e / z_e) f_e + z_e k_e \right)$$

s.t.: $\mathbf{z} = (z_e)_{e \in E}$, $z_e \in \mathbb{R}_+$

$\mathbf{f} = (f_e)_{e \in E}$ is a Wardrop equilibrium



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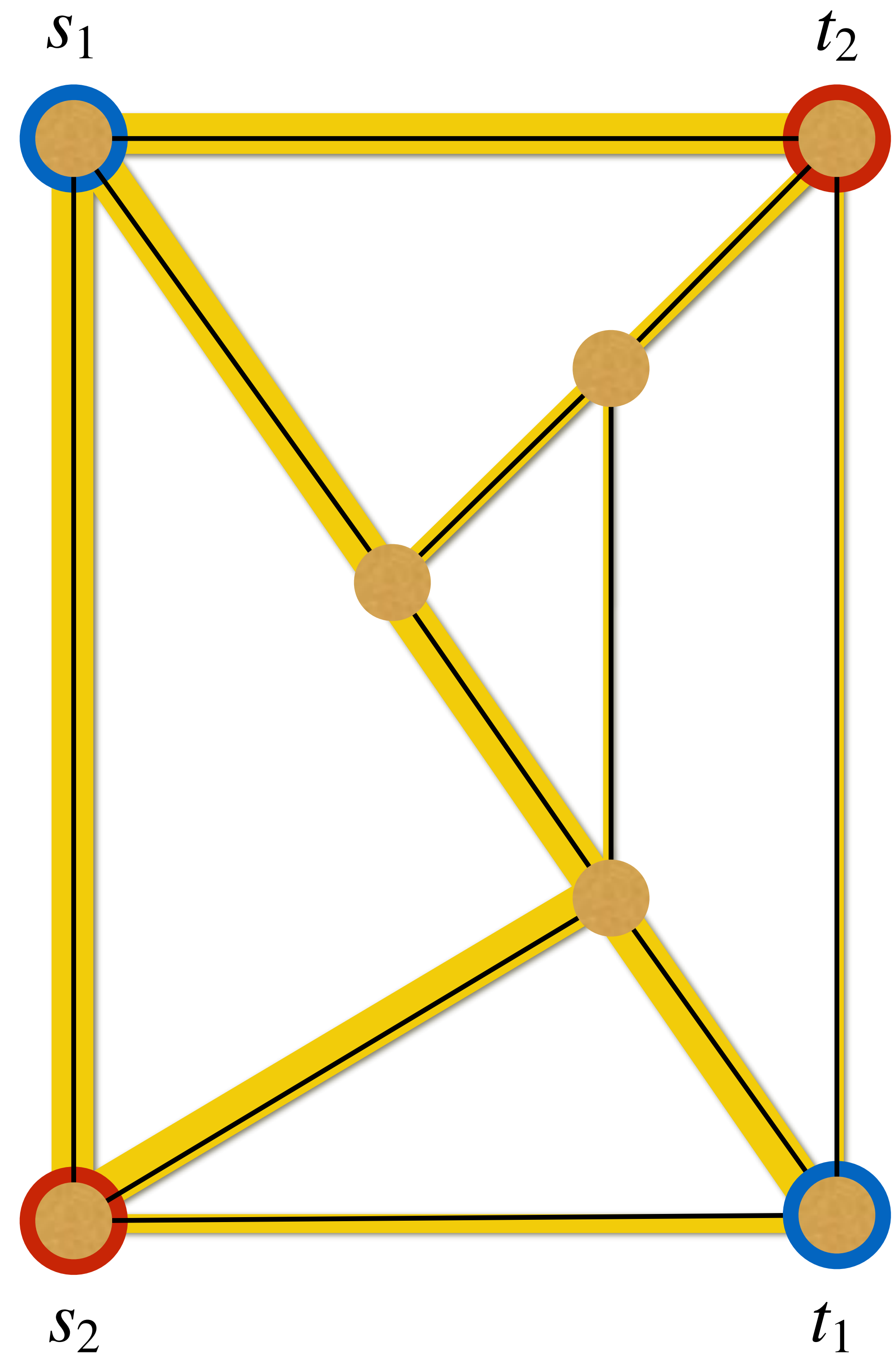

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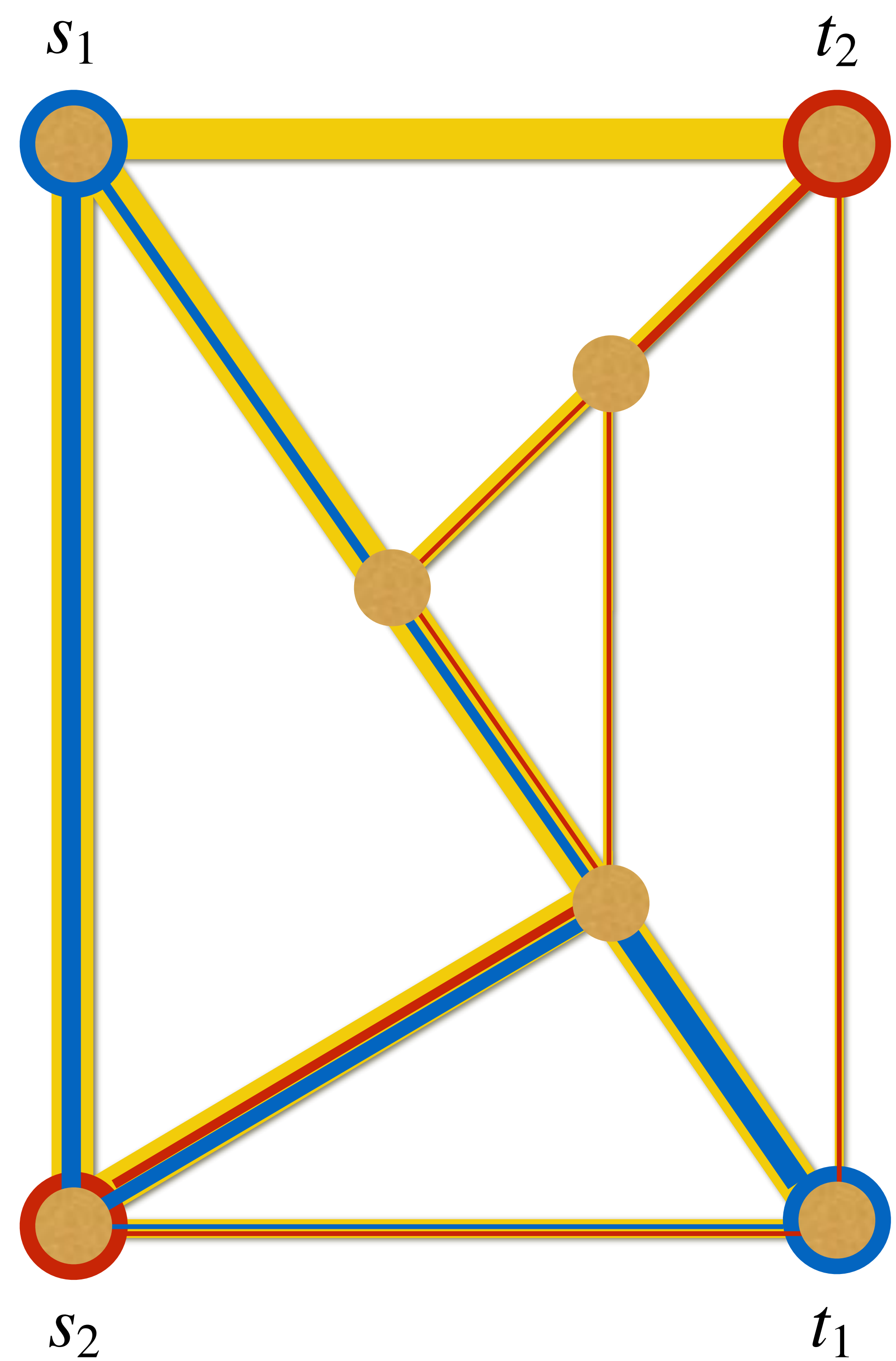

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equilibrium constraints

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$\mathbf{f} \in \operatorname{argmin}_{g\,:\,g\text{ flow}} \sum_{e\in E} \int_0^{g_e} c_e(t/z_e)\,\mathrm{d}t$ — potential function

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$\Leftrightarrow$

$\sum_{e\in E} c_e(f_e/z_e)(f_e - g_e) \leq 0$ for all flows $\mathbf{g}$ — variational inequality

Example

flow f_e

capacity z_e

latency $c_e(f_e/z_e)$

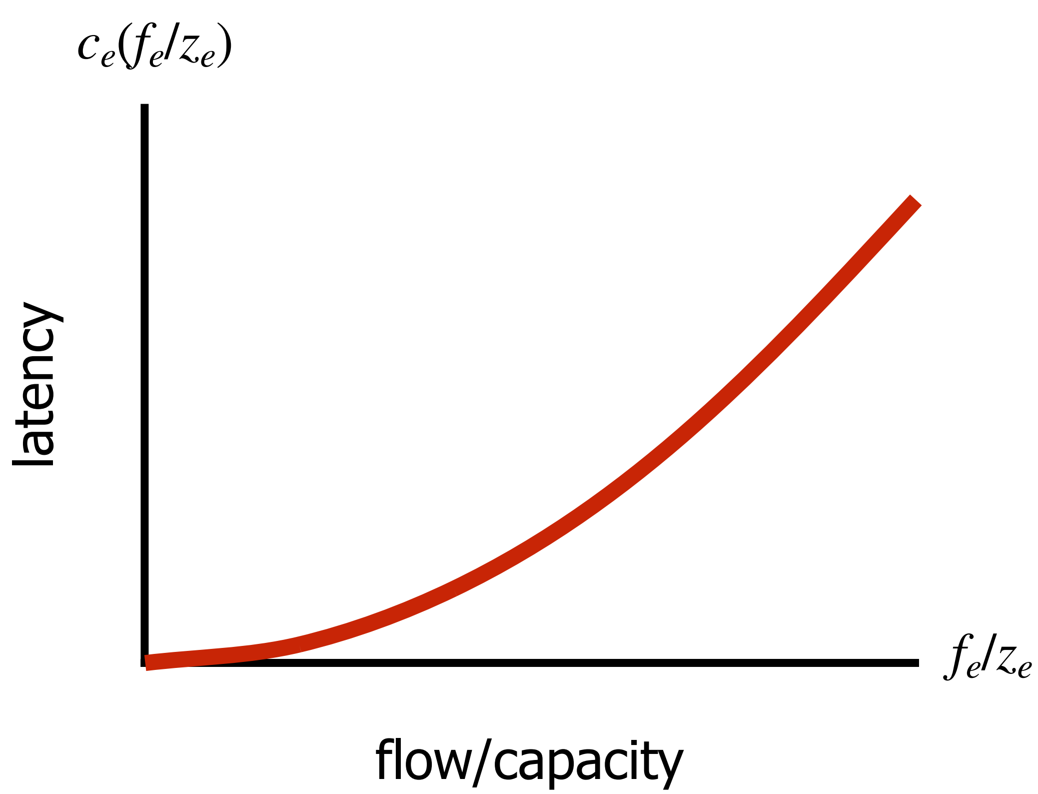
Example

Example

flow f_e capacity z_e latency $c_e(f_e/z_e)$

Example


road e
flow f_e
capacity z_e
latency $c_e(f_e/z_e)$


$c_e(f_e/z_e)$
latency
f_e/z_e
flow/capacity

Example

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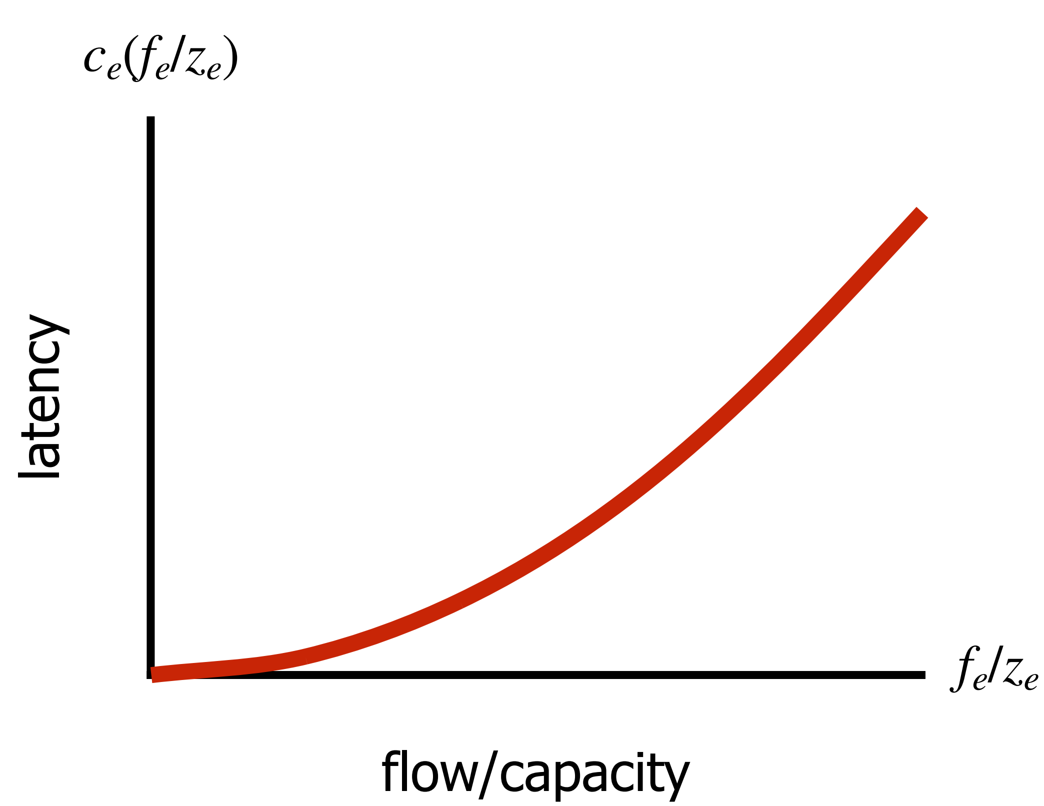

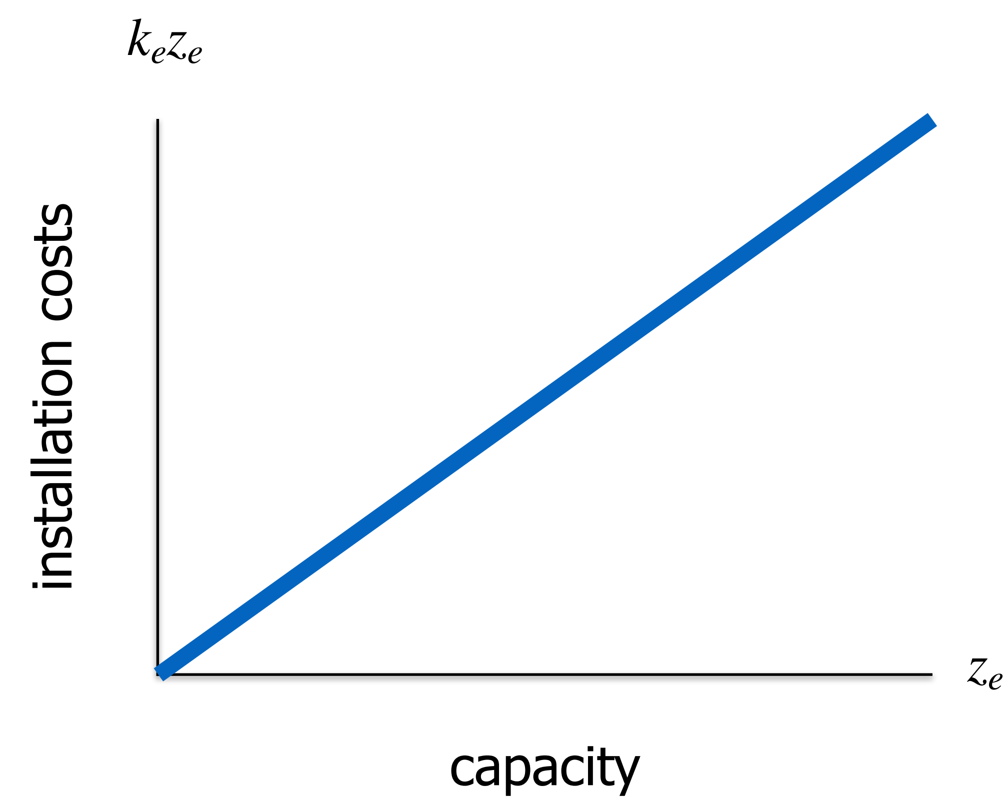

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flow f_e capacity z_e latency $c_e(f_e/z_e)$

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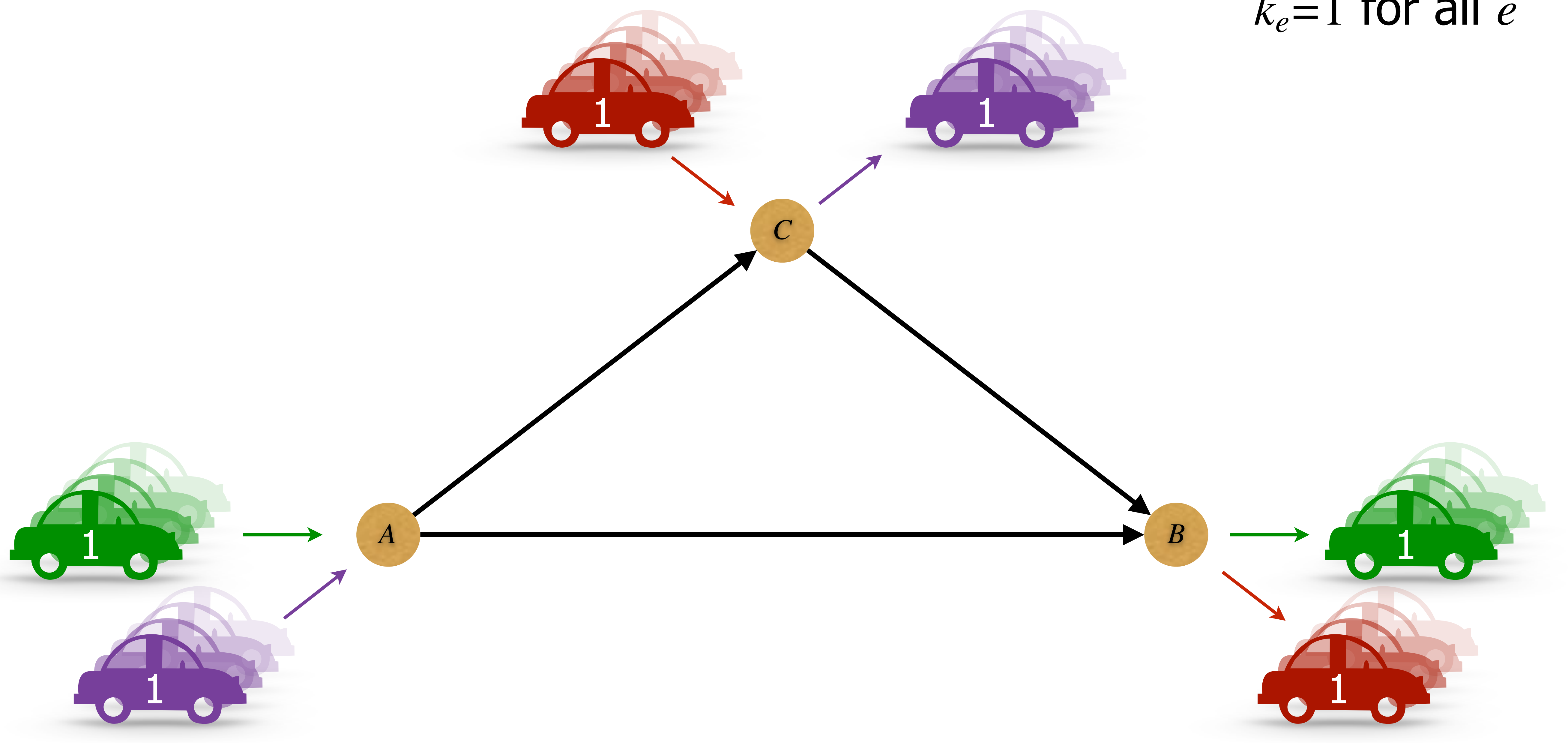
  - optimal total costs for e: $C_e^* = (\sqrt{k_e} + b + \sqrt{k_e})\, f_e$

Example

$k_e = 1$ for all e



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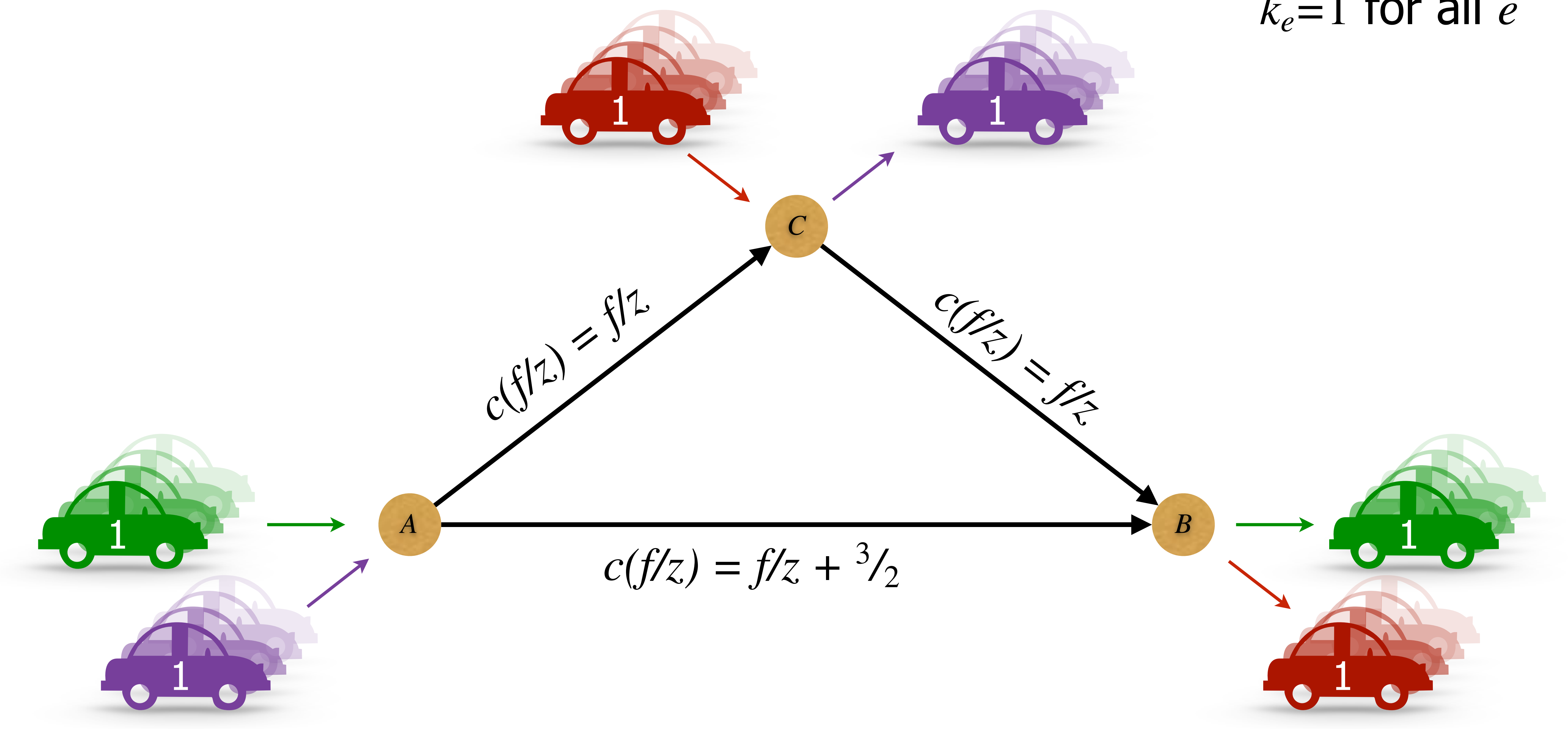

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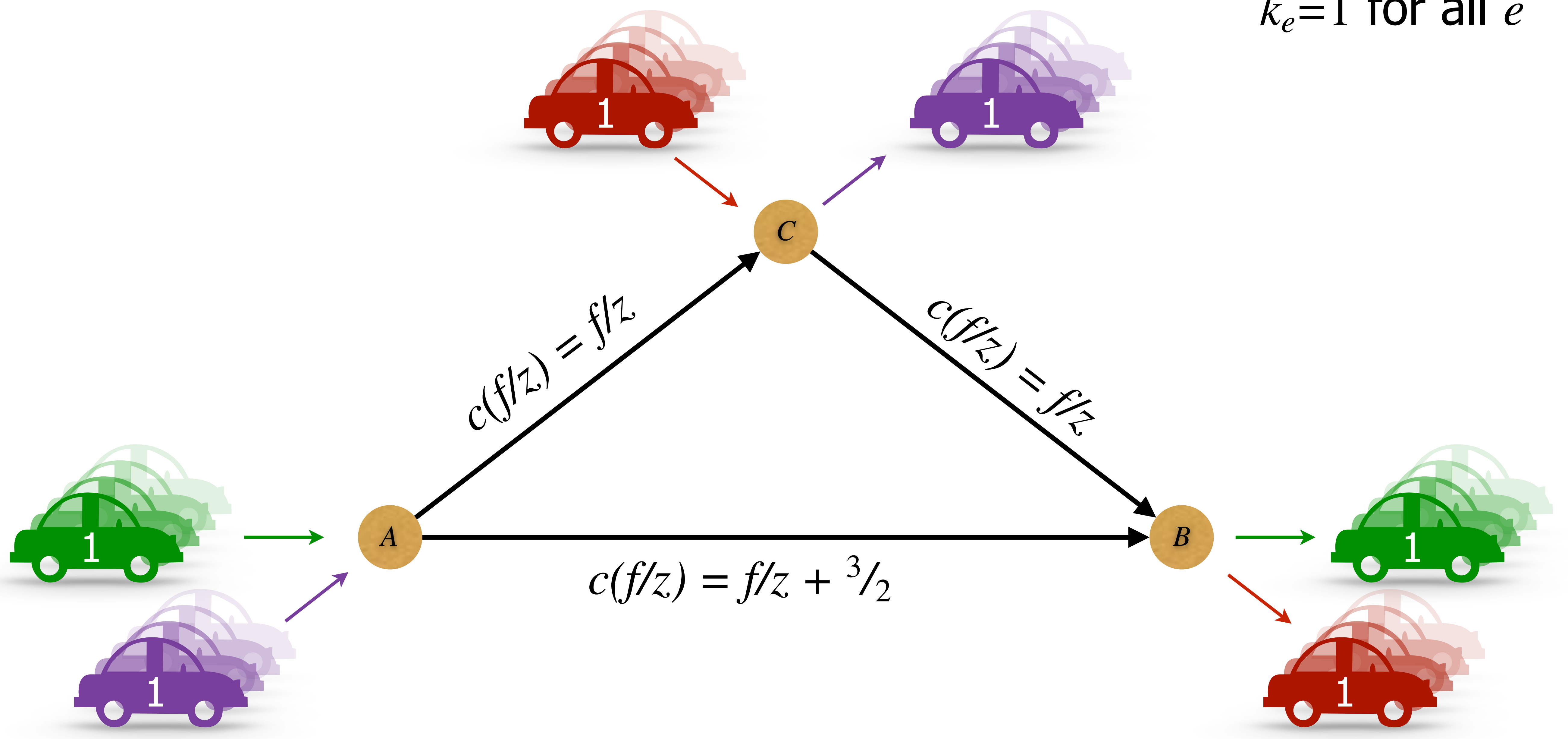

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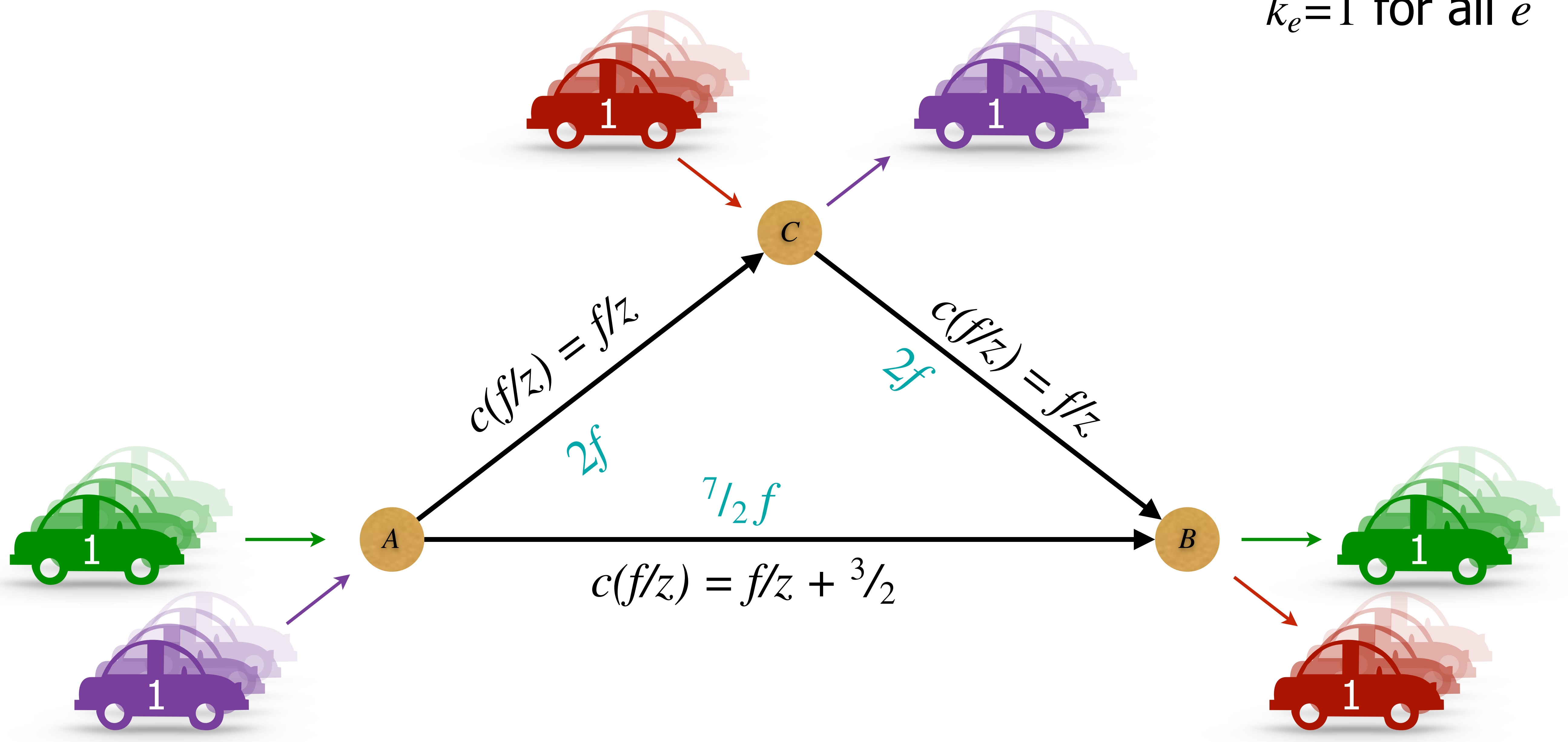

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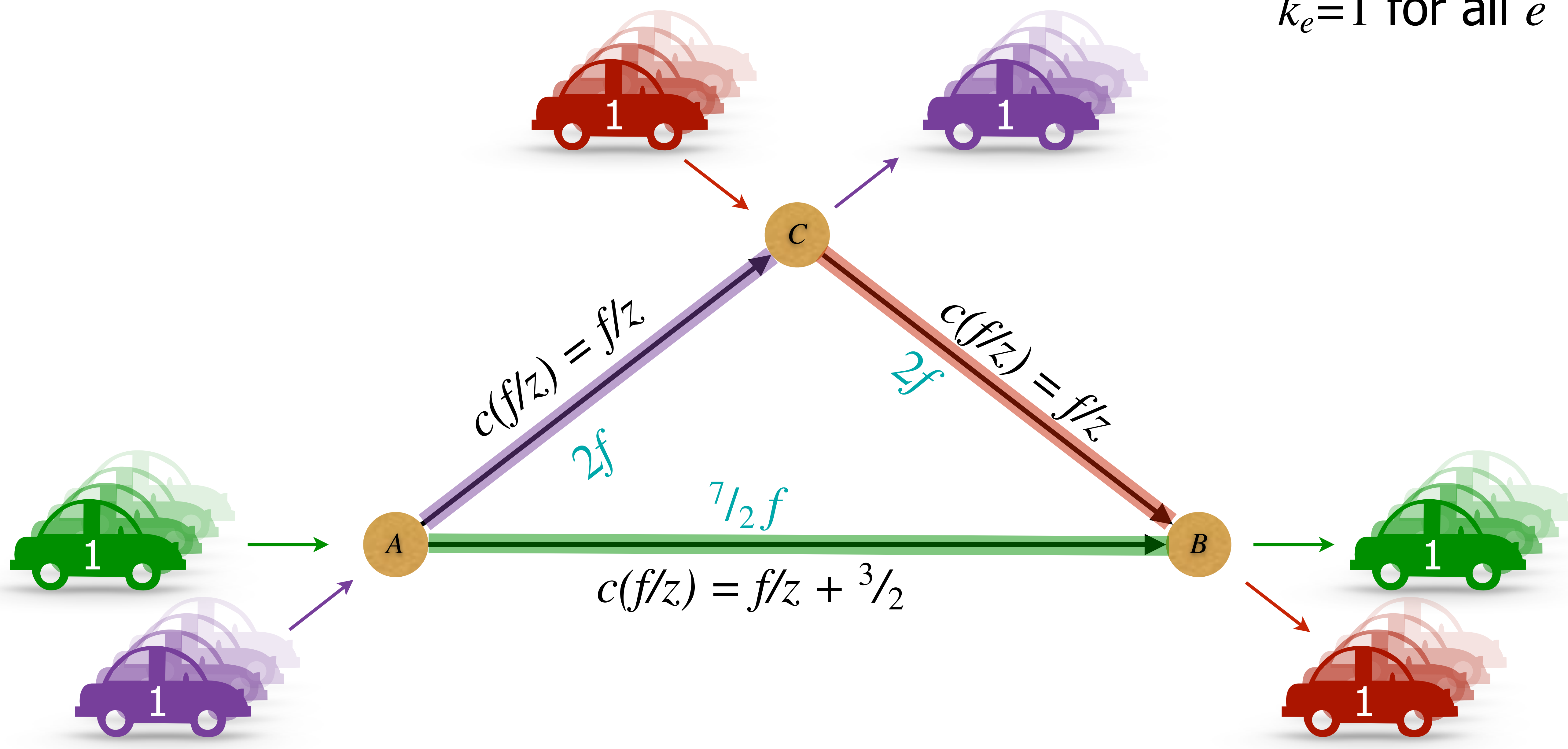

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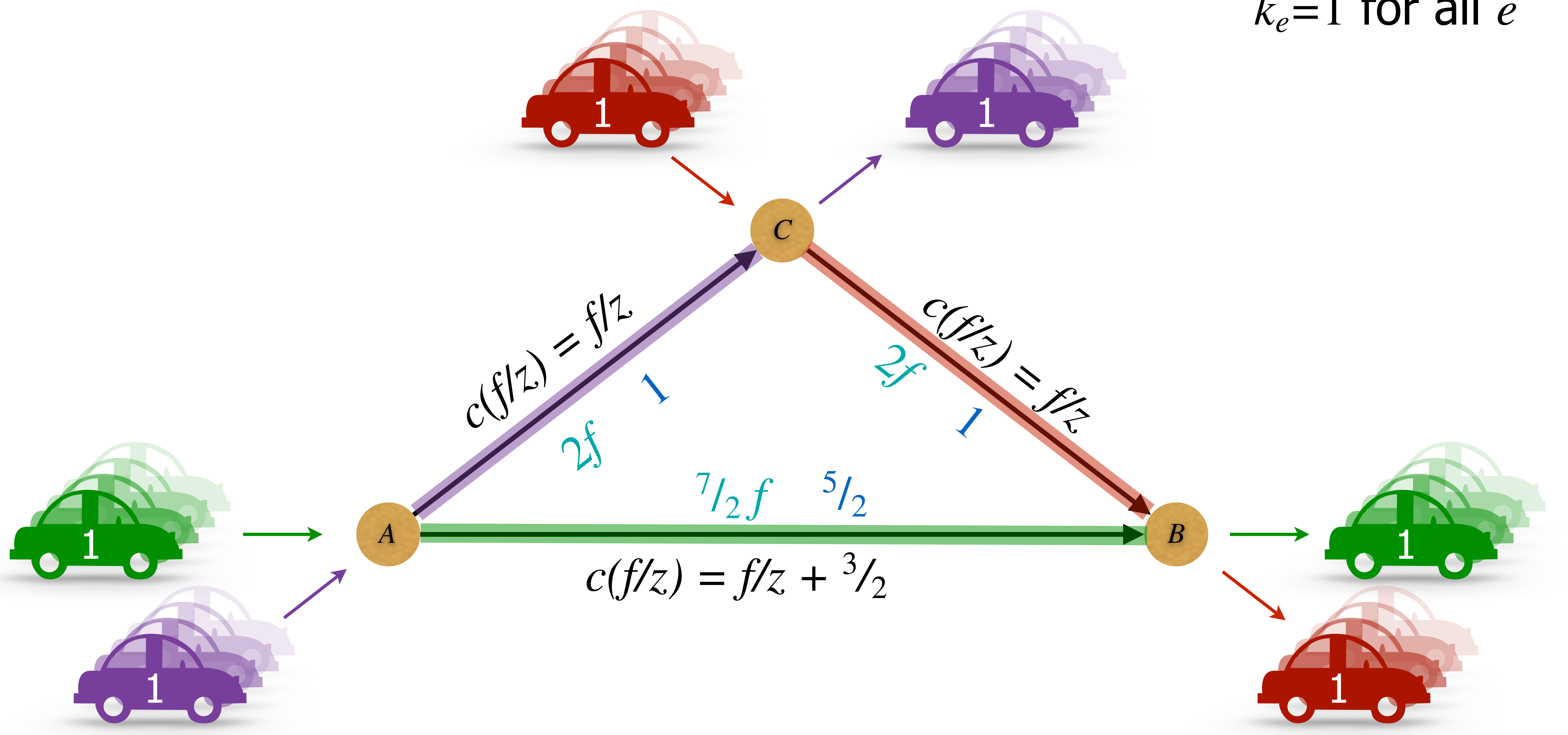

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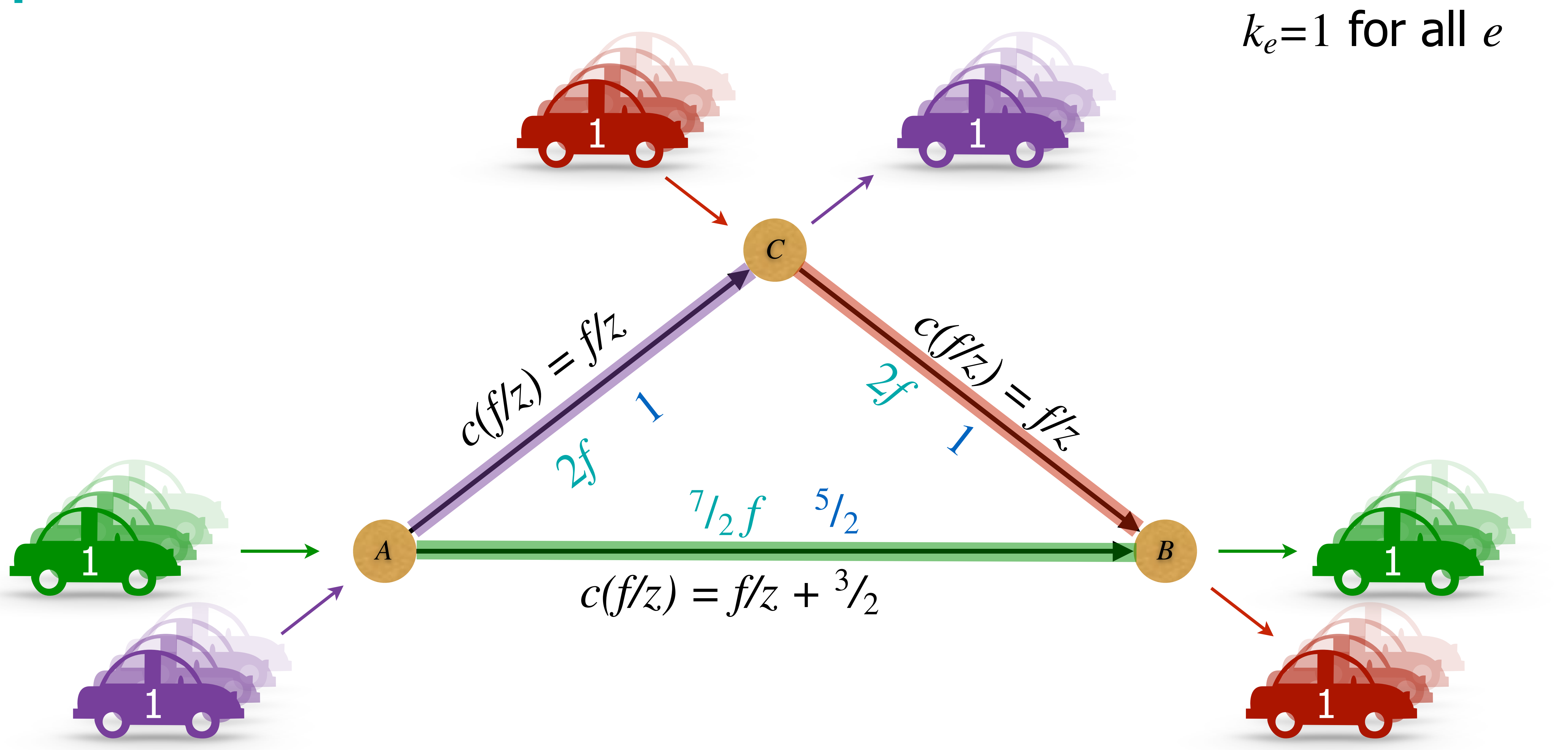

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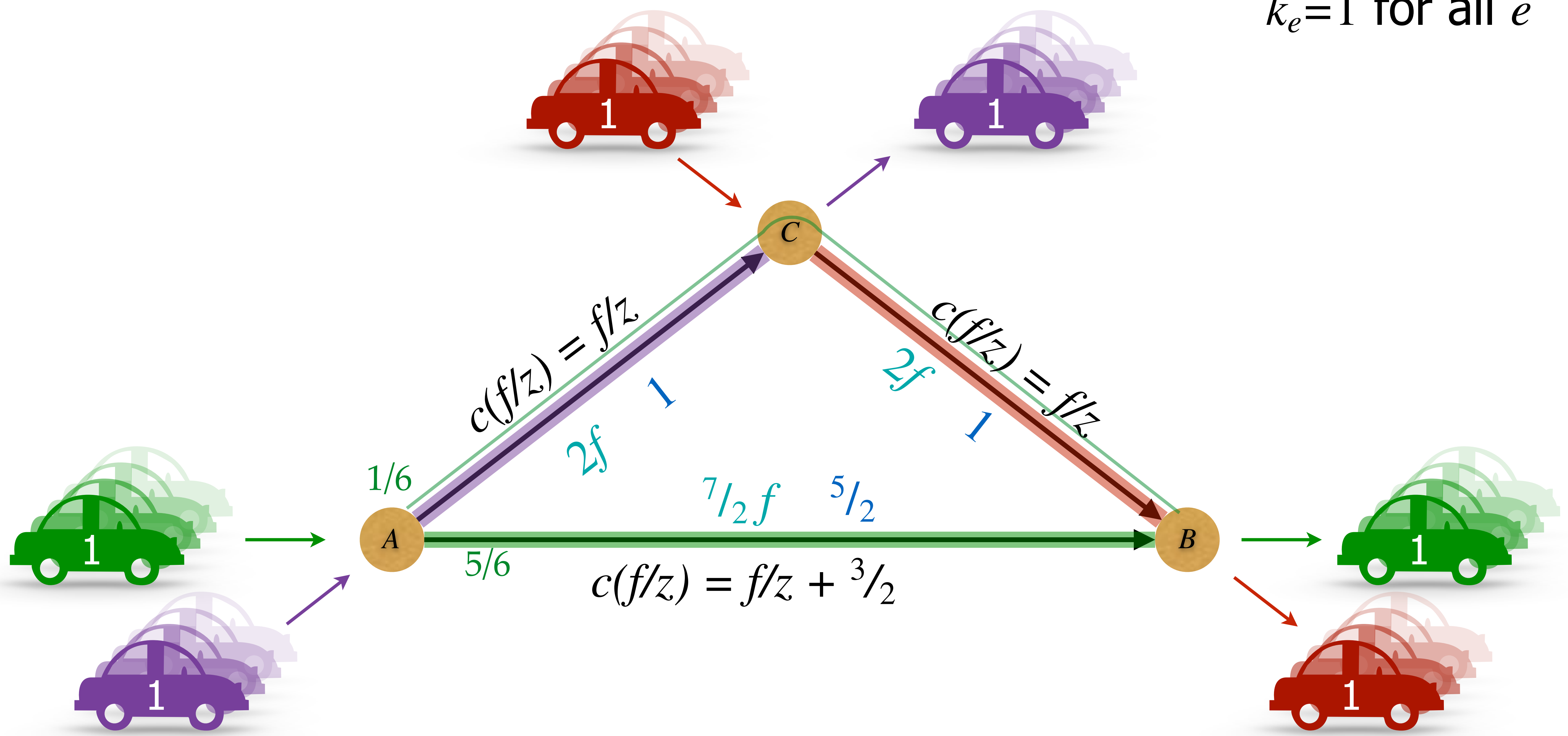

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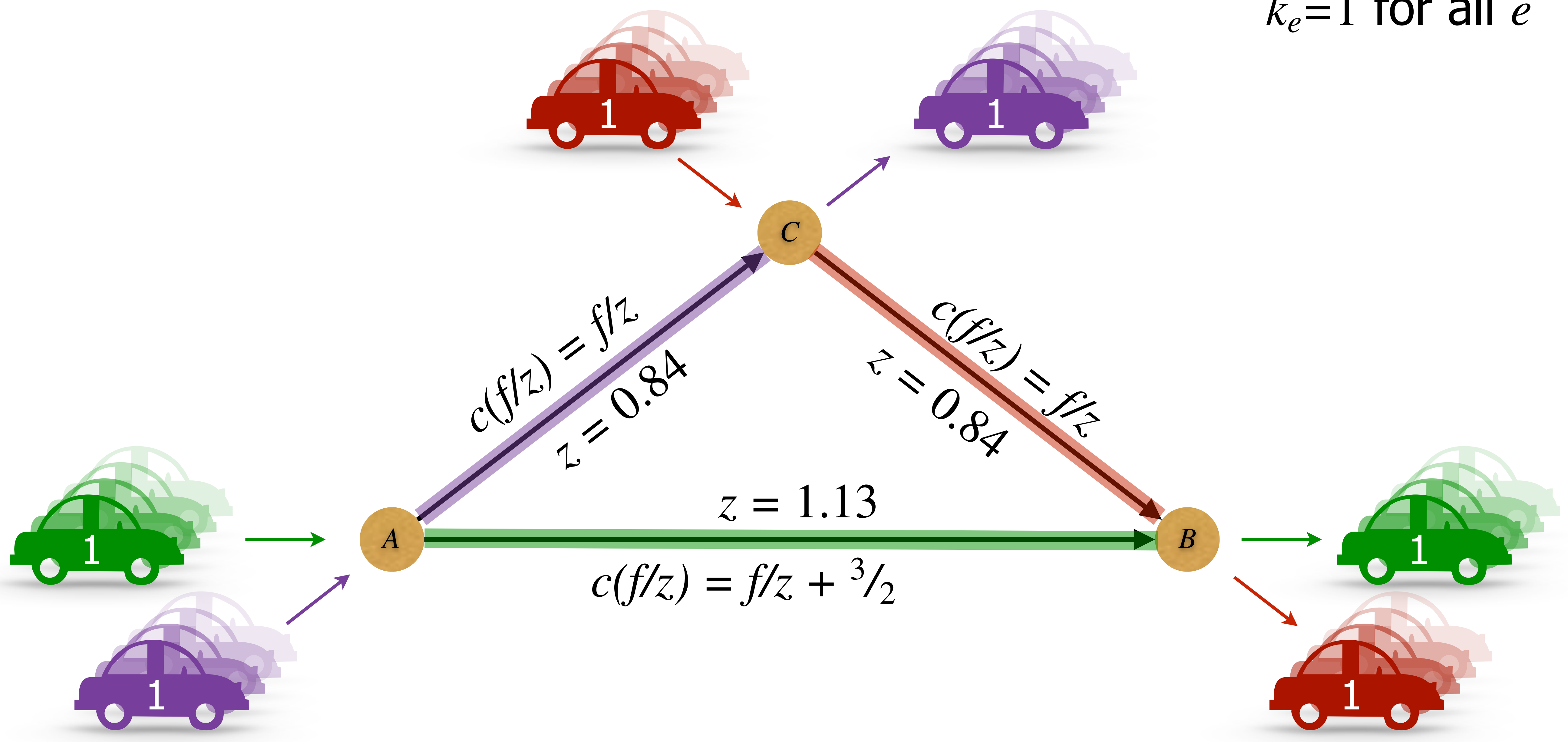

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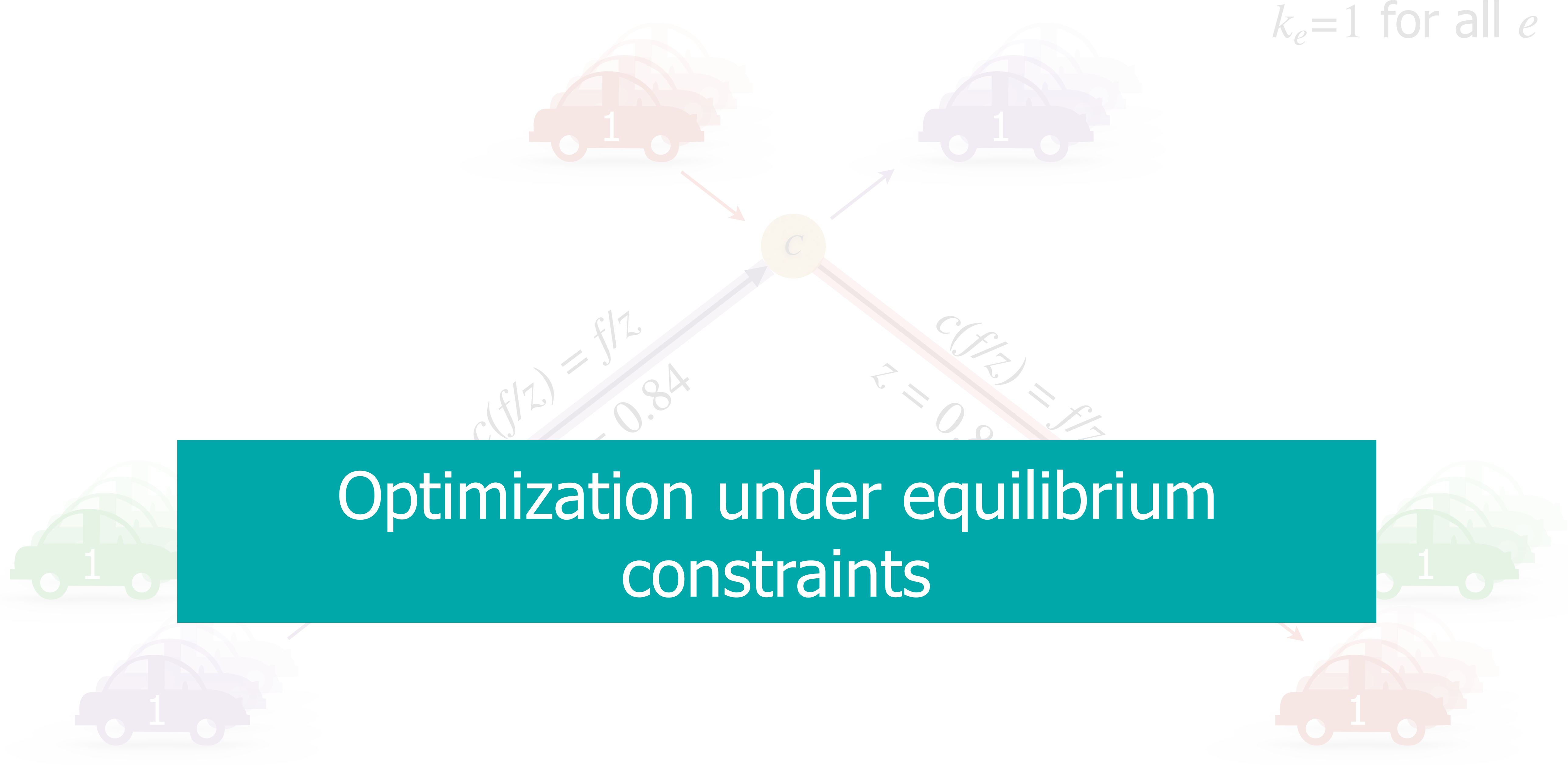

Optimization under equilibrium constraints

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Previous work

- "one of the most important, difficult and challenging problems in transport" [Yang, Bell '98]
- various heuristics e.g. [Dafermos '69][Dantzig et al. '79]
- approximation algorithm [Marcotte '85]
  - $5/4$-approximation for affine costs $c(f/z) = a + b(f/z)$
  - closed formula for monomials $c(f/z) = a + b(f/z)^d$, converges to 2 as $d \to \infty$

Previous work   Our contribution

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→ approximation algorithm with improved guarantee

Complexity

Complexity

Theorem [Gairing, Harks & K., '14]

The CNDP is **APX**-hard.

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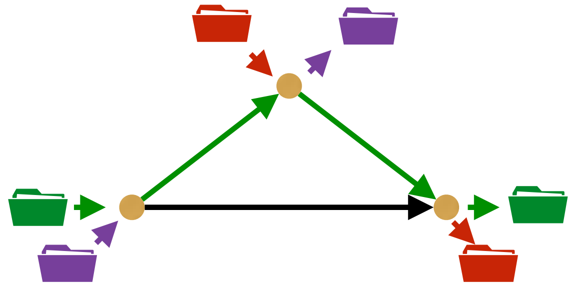

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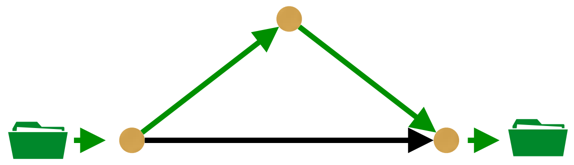

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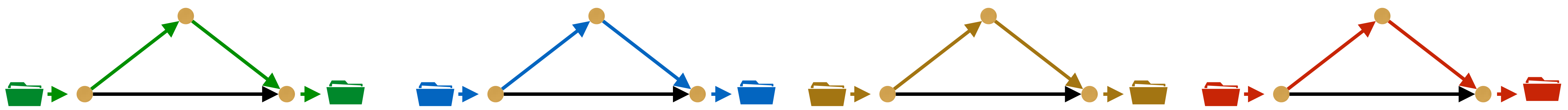

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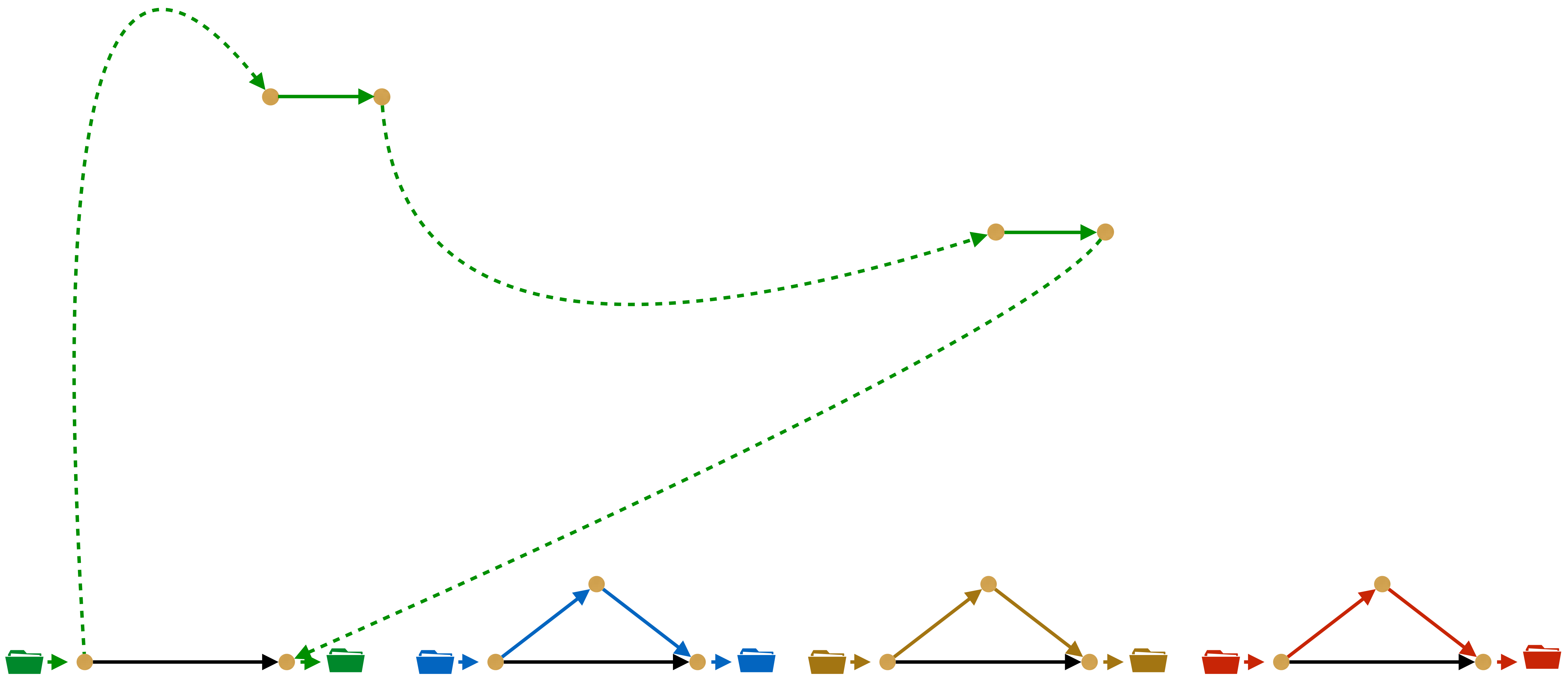

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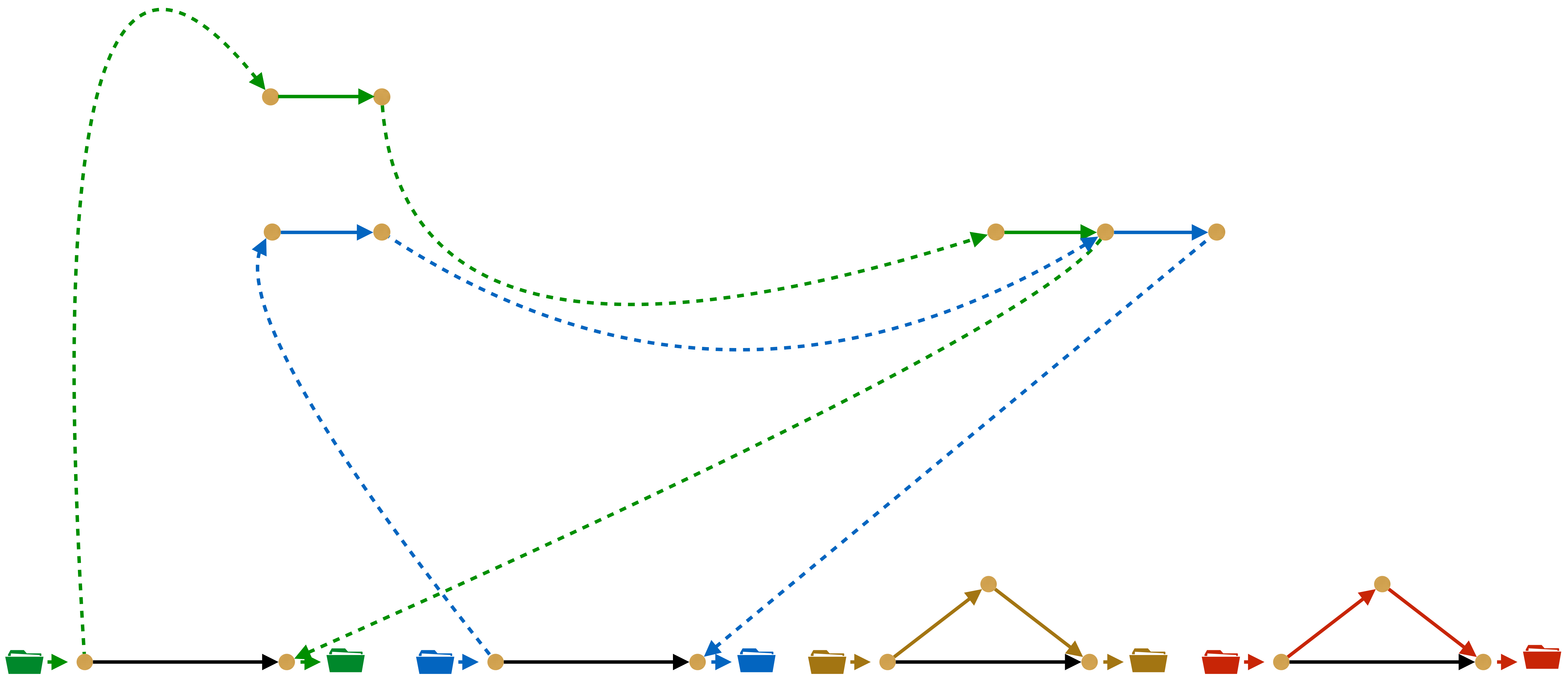

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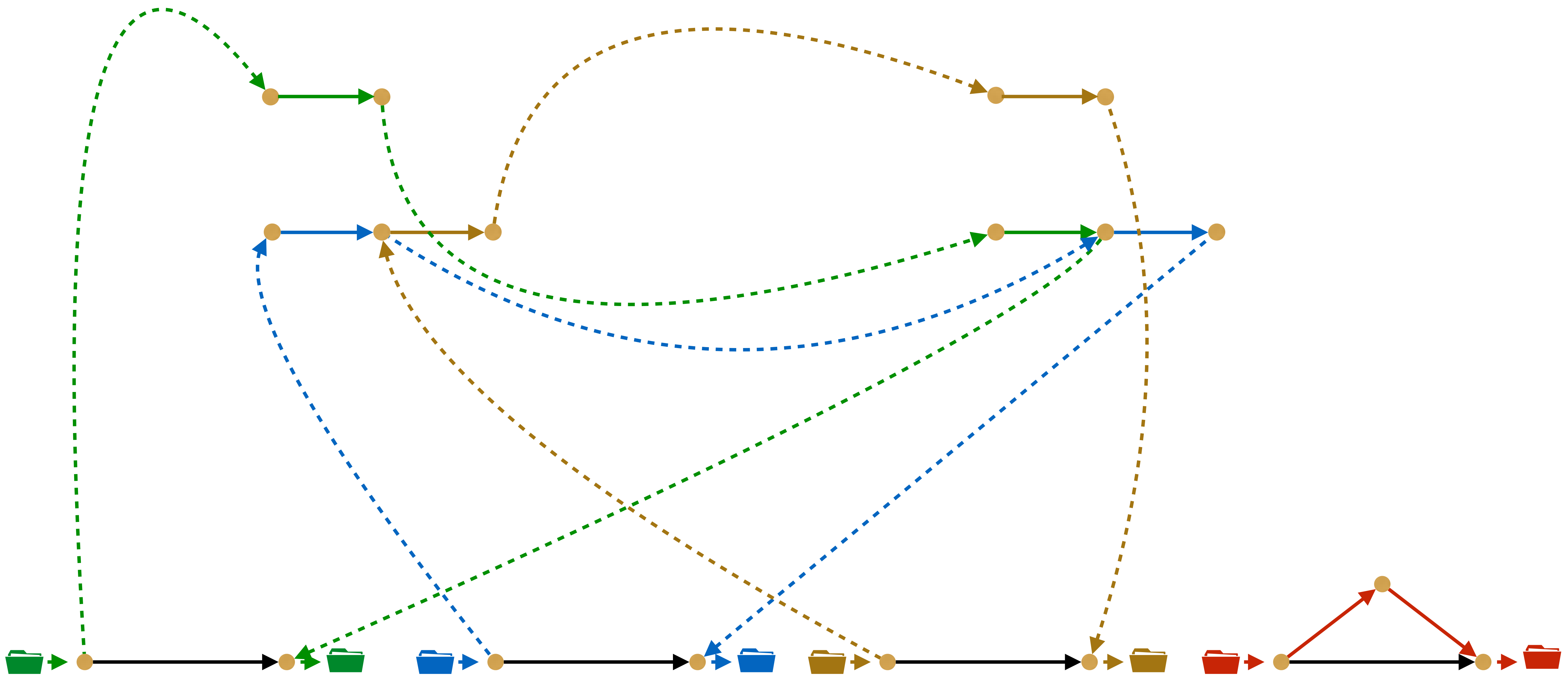

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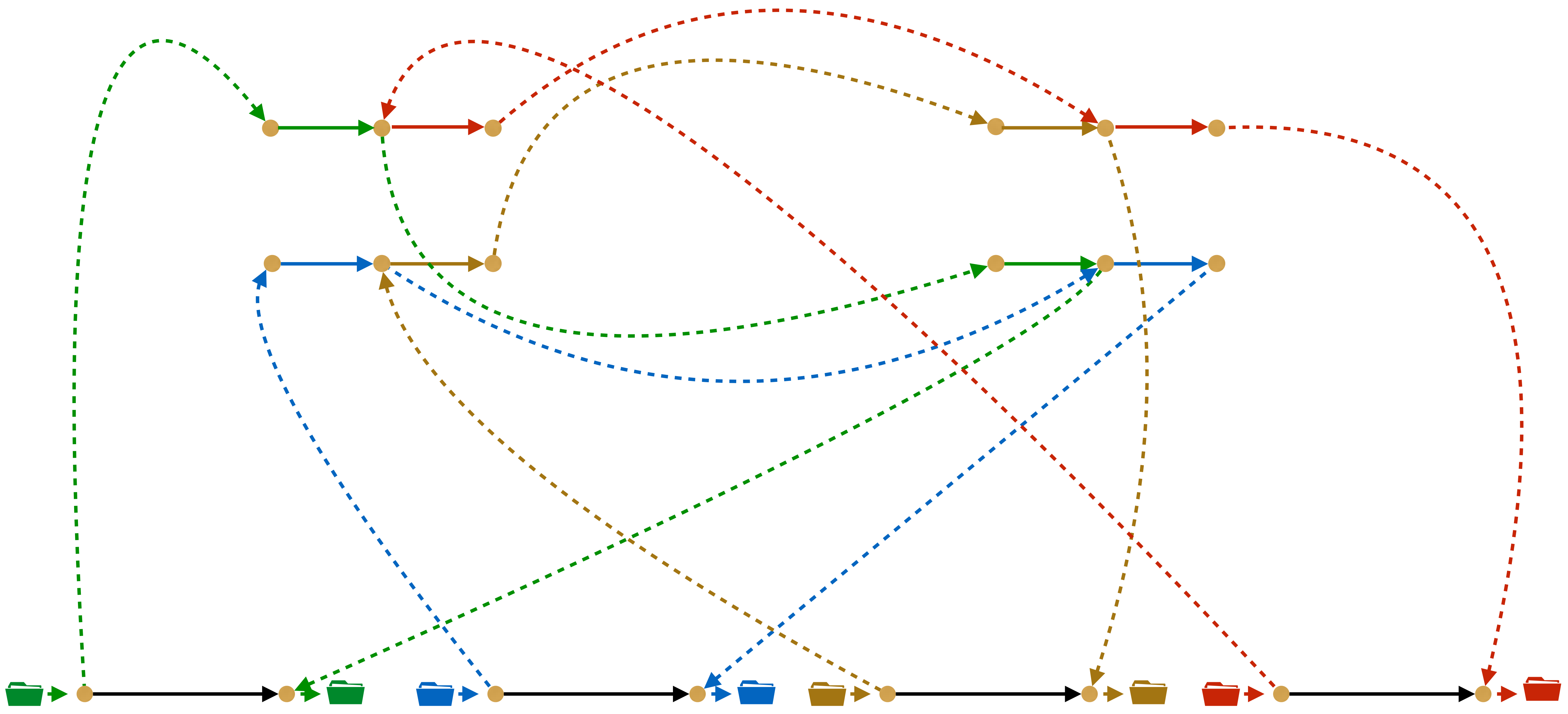

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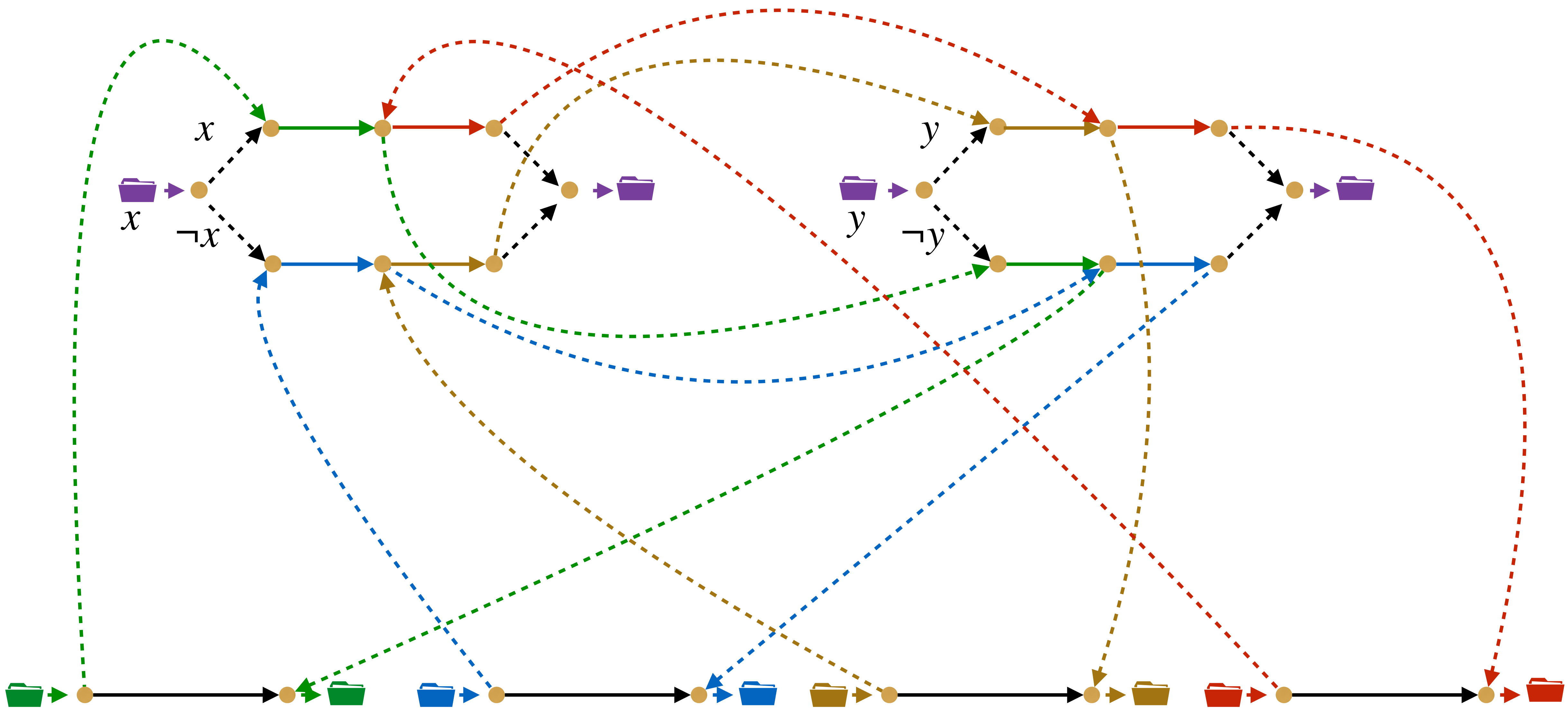

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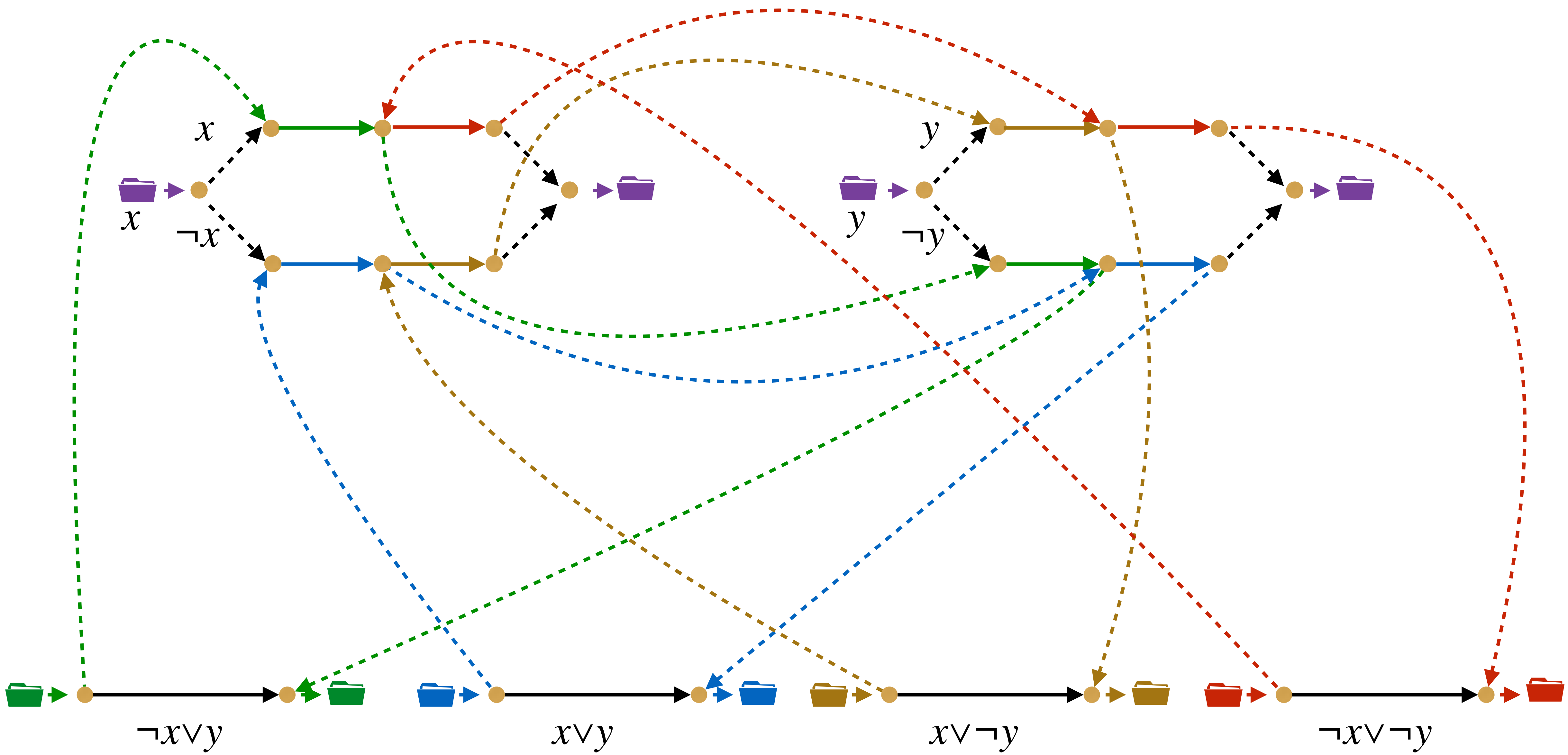


Approximation

The relaxation

Continuous network design problem — Relaxation

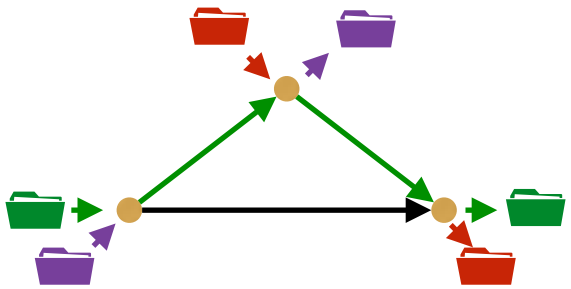
$$\min_{\mathbf{f},\, \mathbf{z}} \sum_{e \in E} \left(c_e(f_e / z_e) f_e + z_e k_e \right)$$

$$\text{s.t.: } \mathbf{z} = (z_e)_{e \in E},\, z_e \in \mathbb{R}_+$$

$$\mathbf{f} = (f_e)_{e \in E} \text{ is a flow}$$

Lemma [Marcotte '85; Harks, Gairing, K., '14]

The relaxation can be solved in polynomial time.



Quality of the relaxation

- Price of Anarchy (PoA) = $\dfrac{\text{Routing Cost of Wardrop equilibrium}}{\text{Routing cost of optimum}}$

Theorem [Roughgarden, Tardos ’02; Correa at al. ‘04]

$\mathrm{PoA} \leq \frac{1}{1-\mu}$ where $\mu = \sup_{c \in \mathscr{C},\, x \geq 0} \max_{\gamma \in [0,1]} \gamma \left(1 - \frac{c(\gamma \cdot x)}{c(x)}\right).$

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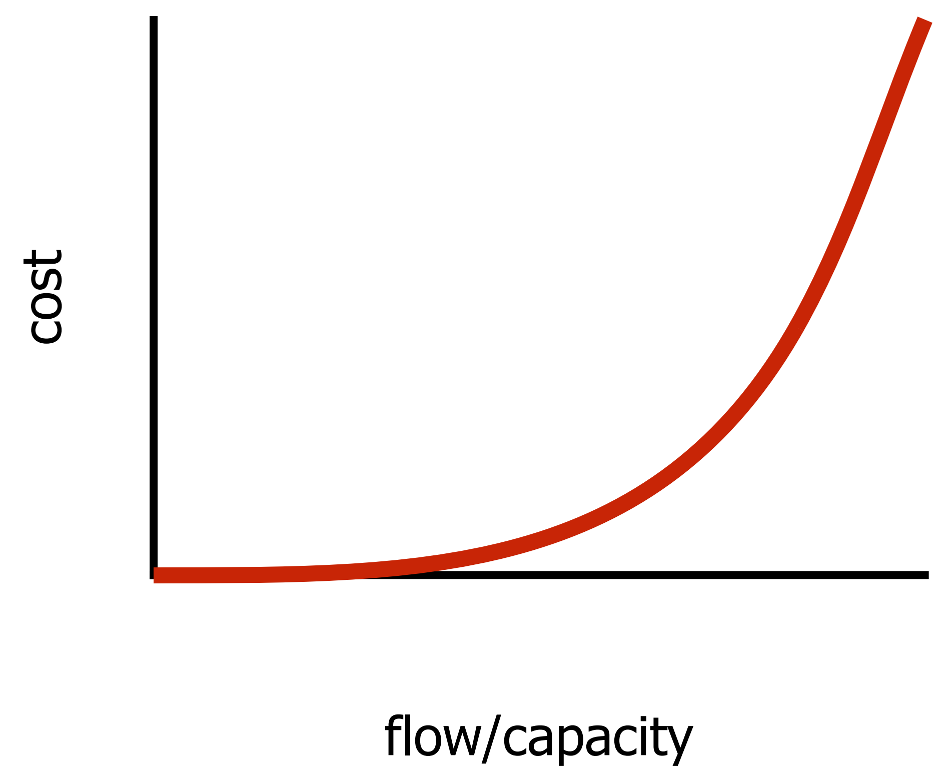

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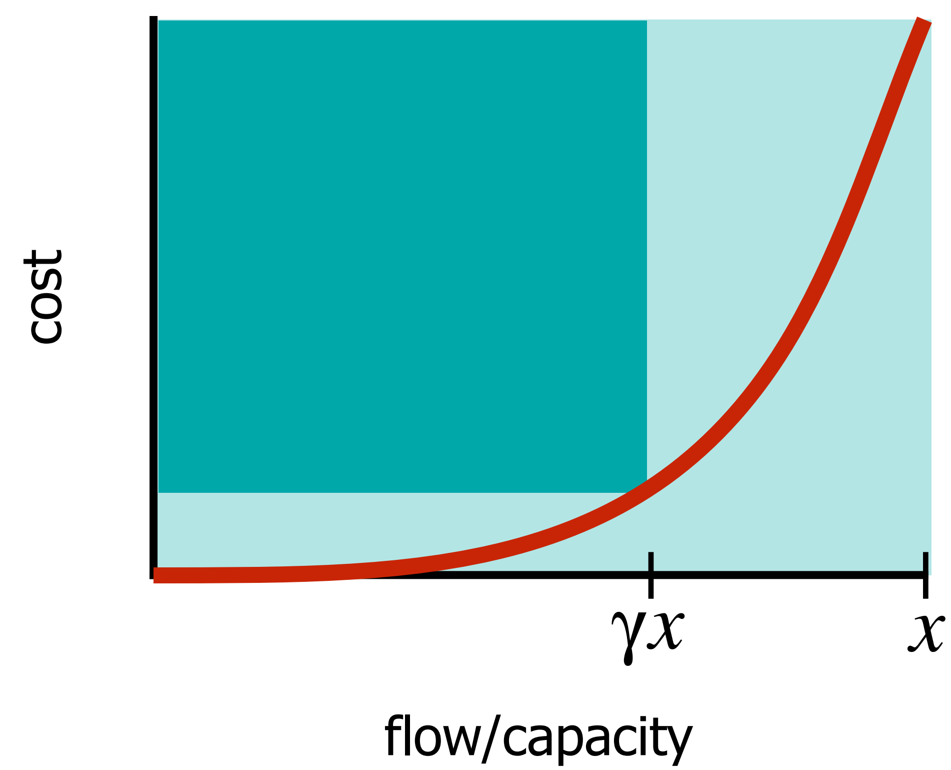

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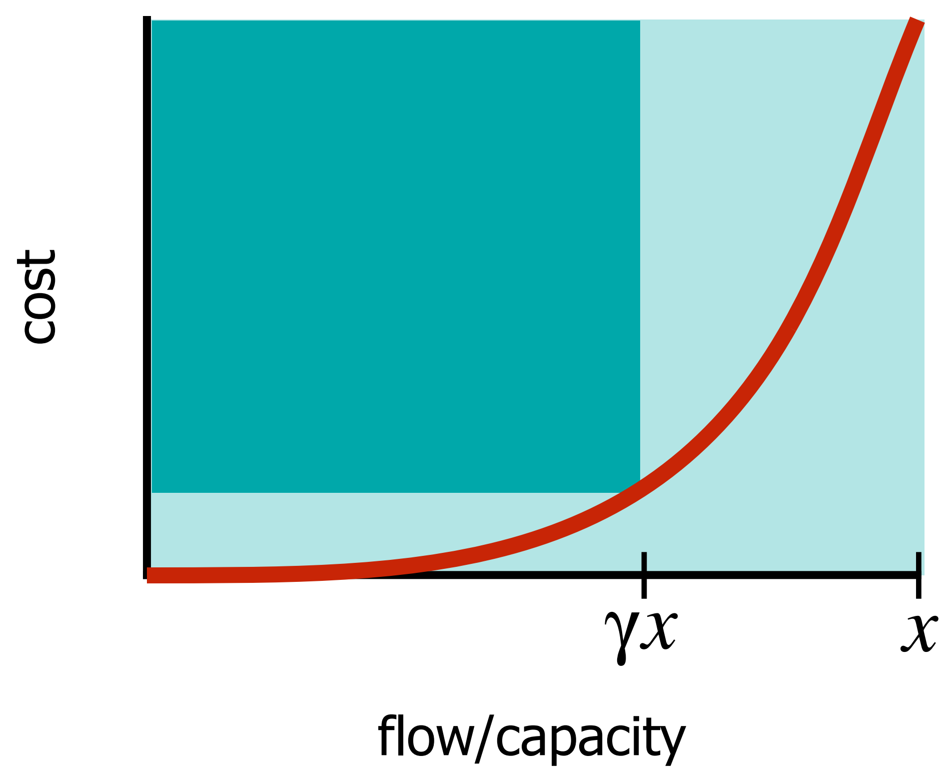

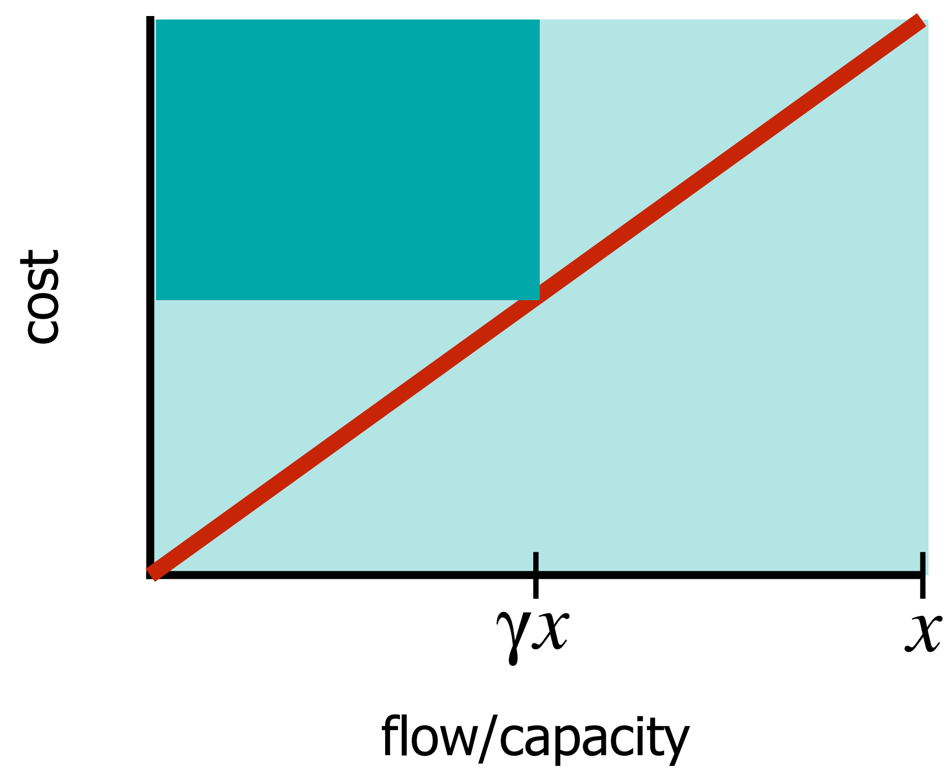

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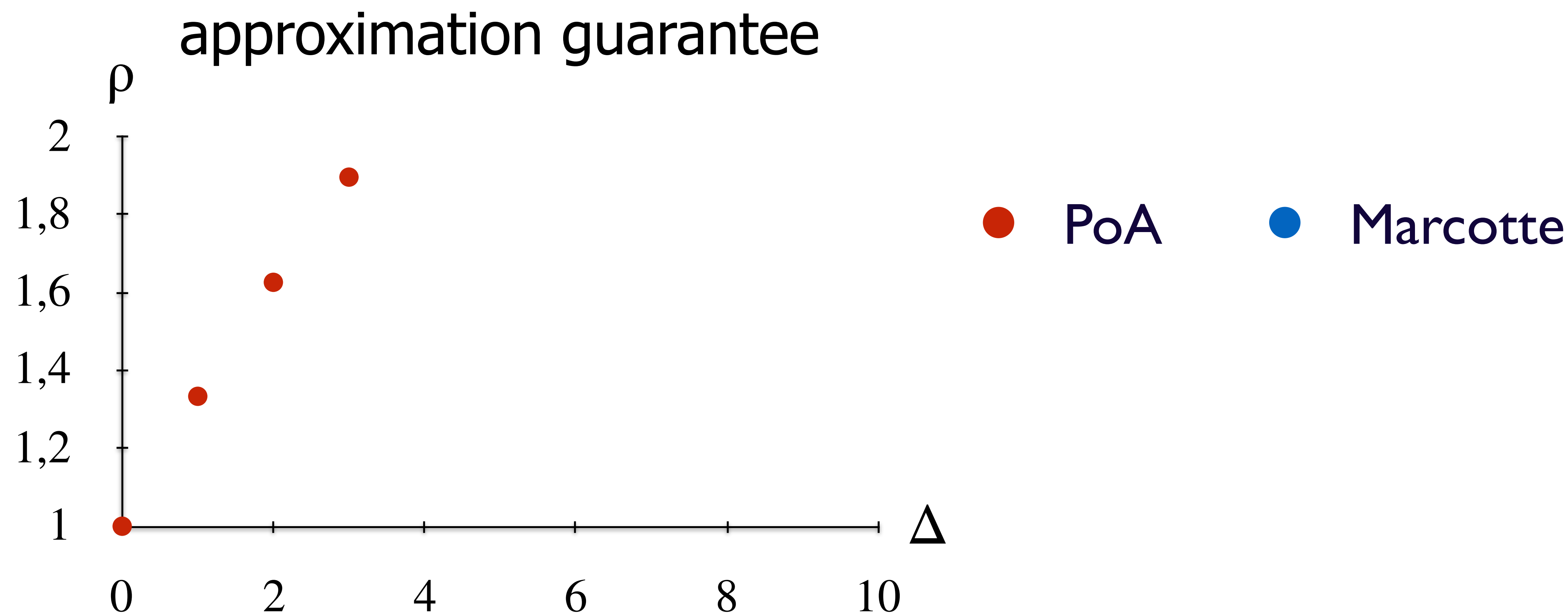

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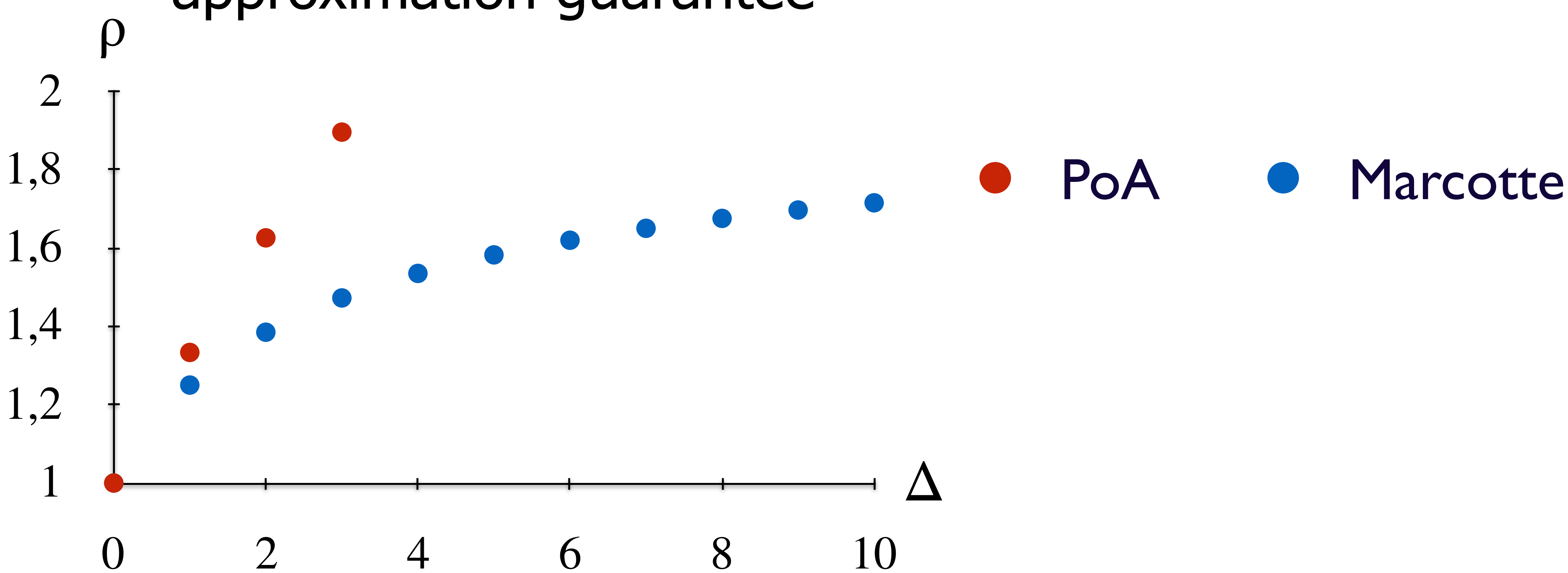
approximation guarantee



Our contribution

Theorem

[Gairing, Harks, K., '14]

There is a ρ-approximation with

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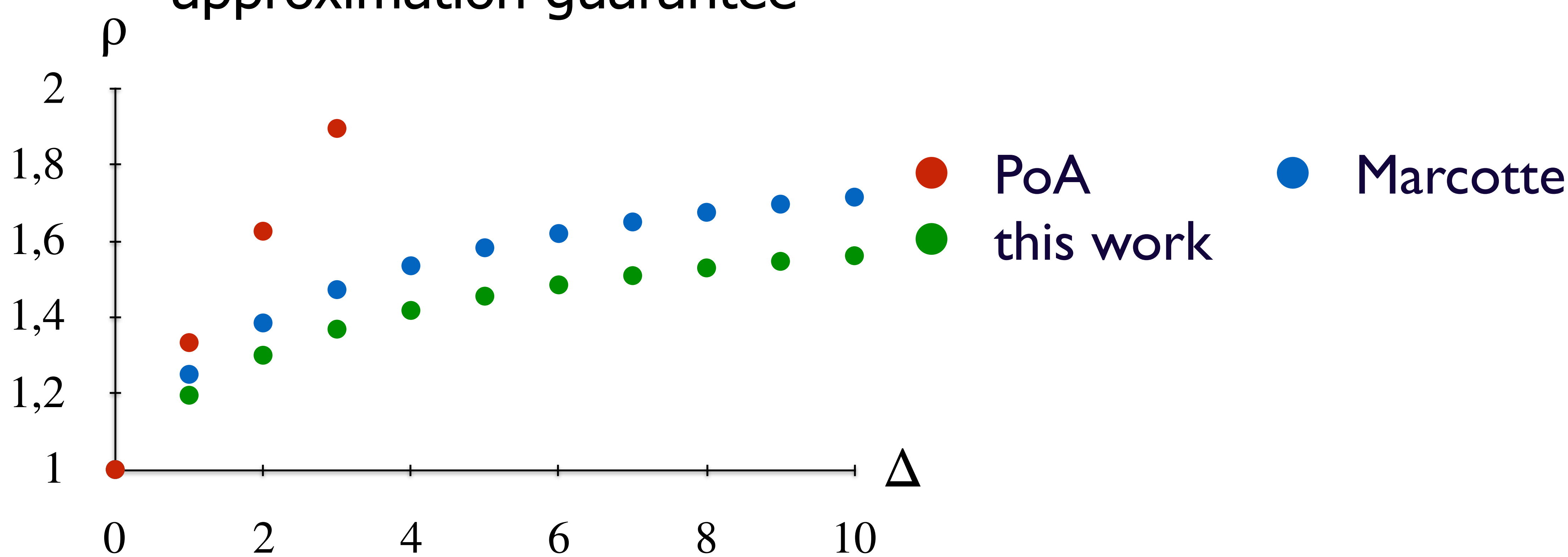

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- Prof: $\mu \leq 1$ and $\gamma \leq 1$ for all $\mathscr{C}$, ρ non-decreasing in μ and γ.

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Input: $G = (N,E)$, (ce)

Output:

- Compute μ,γ
- $(\mathbf{z}, \mathbf{f}) \leftarrow$ Solution of the relaxation
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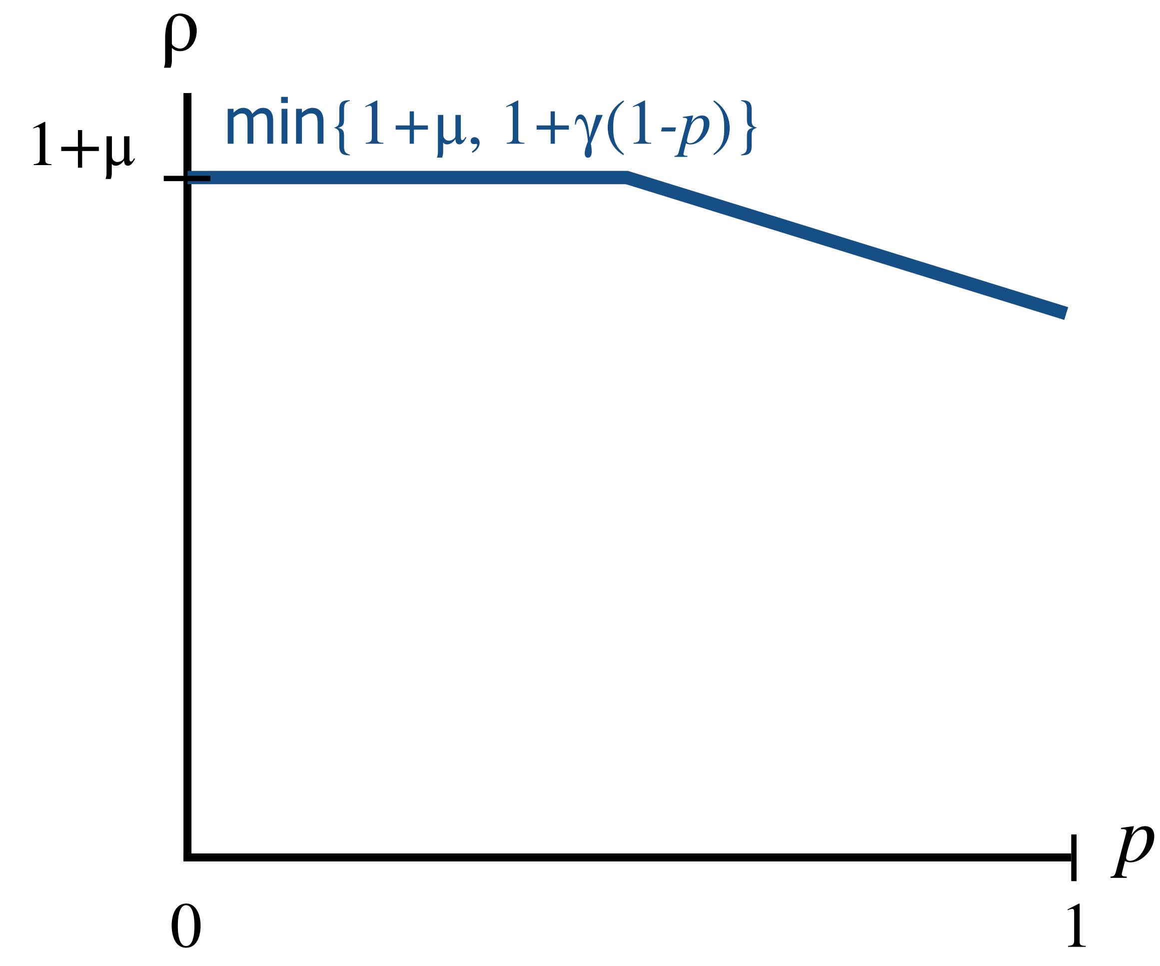

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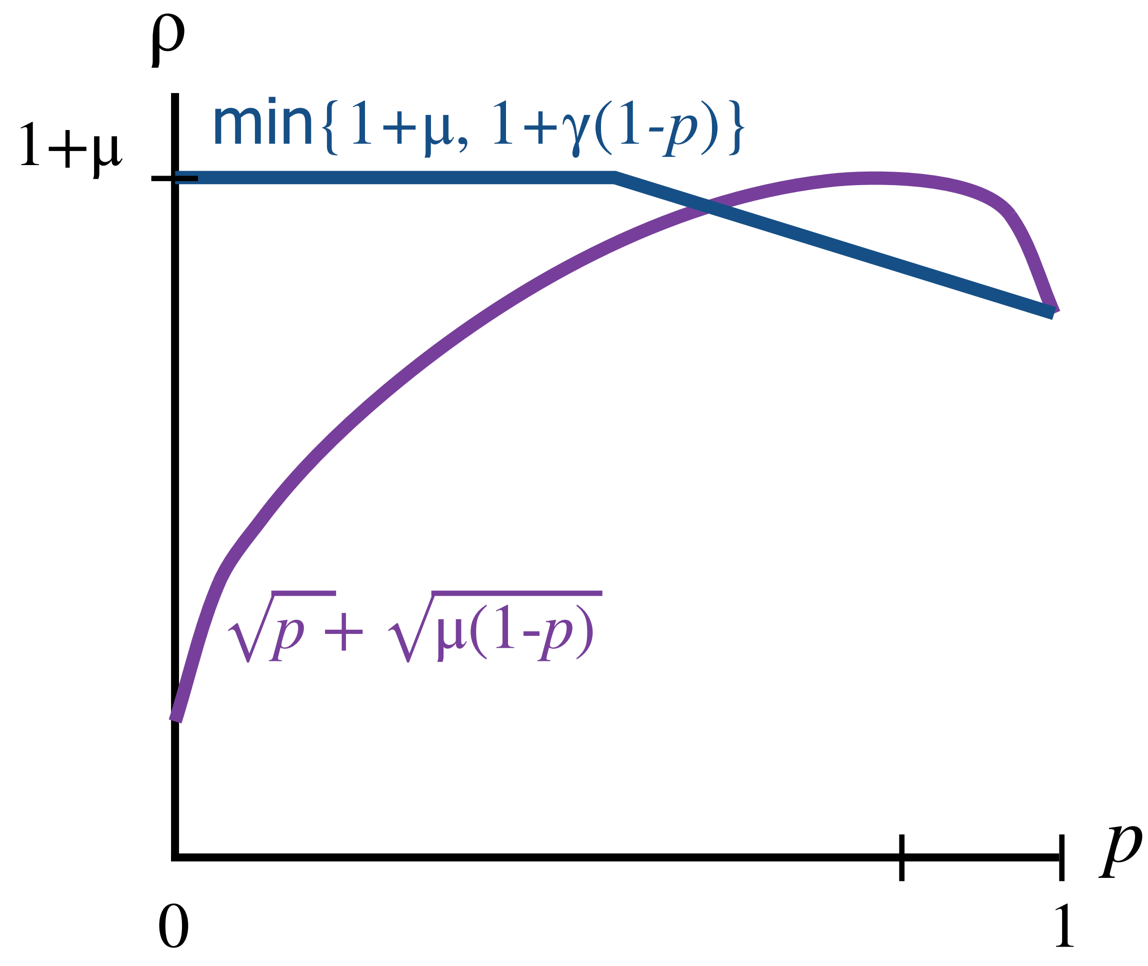

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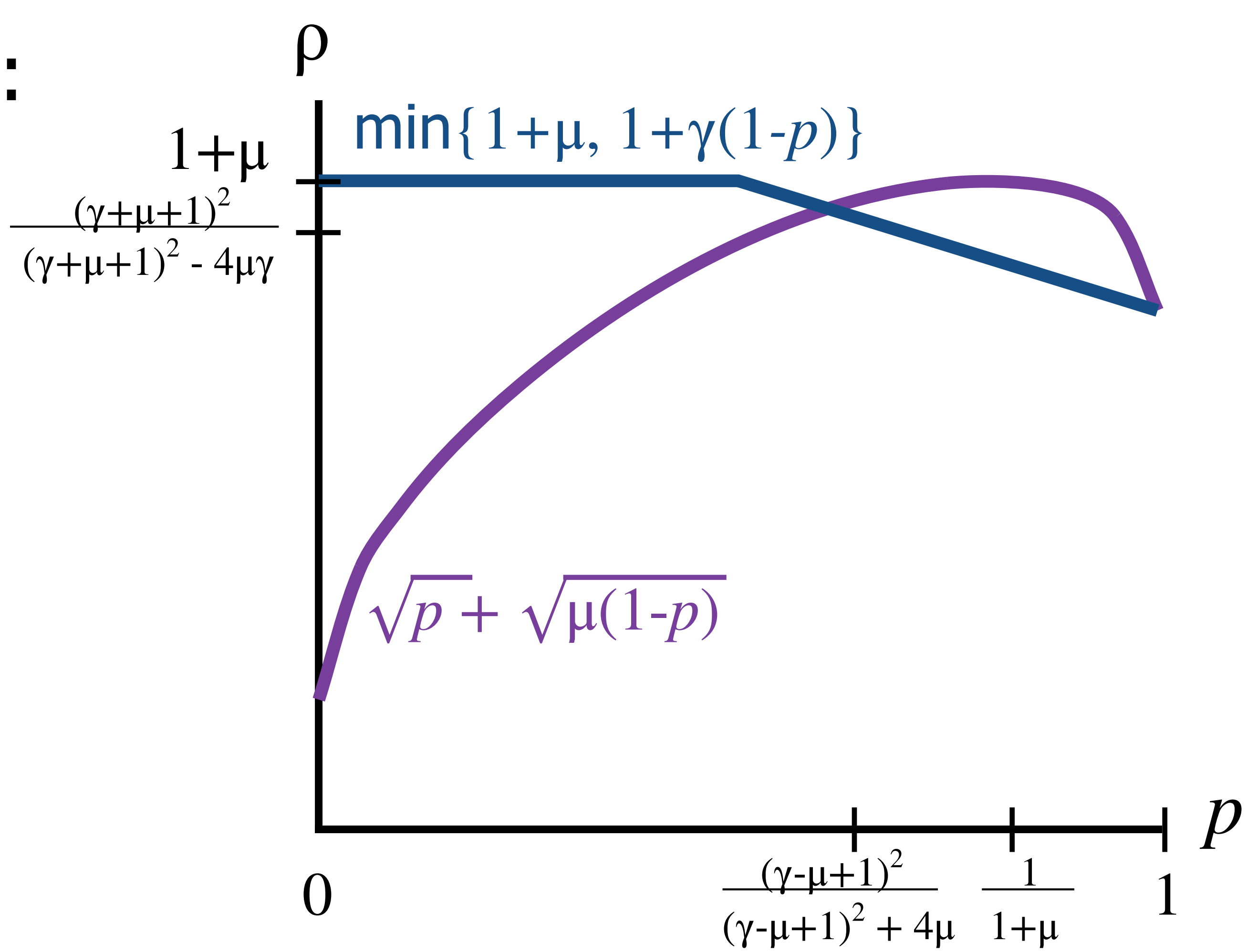

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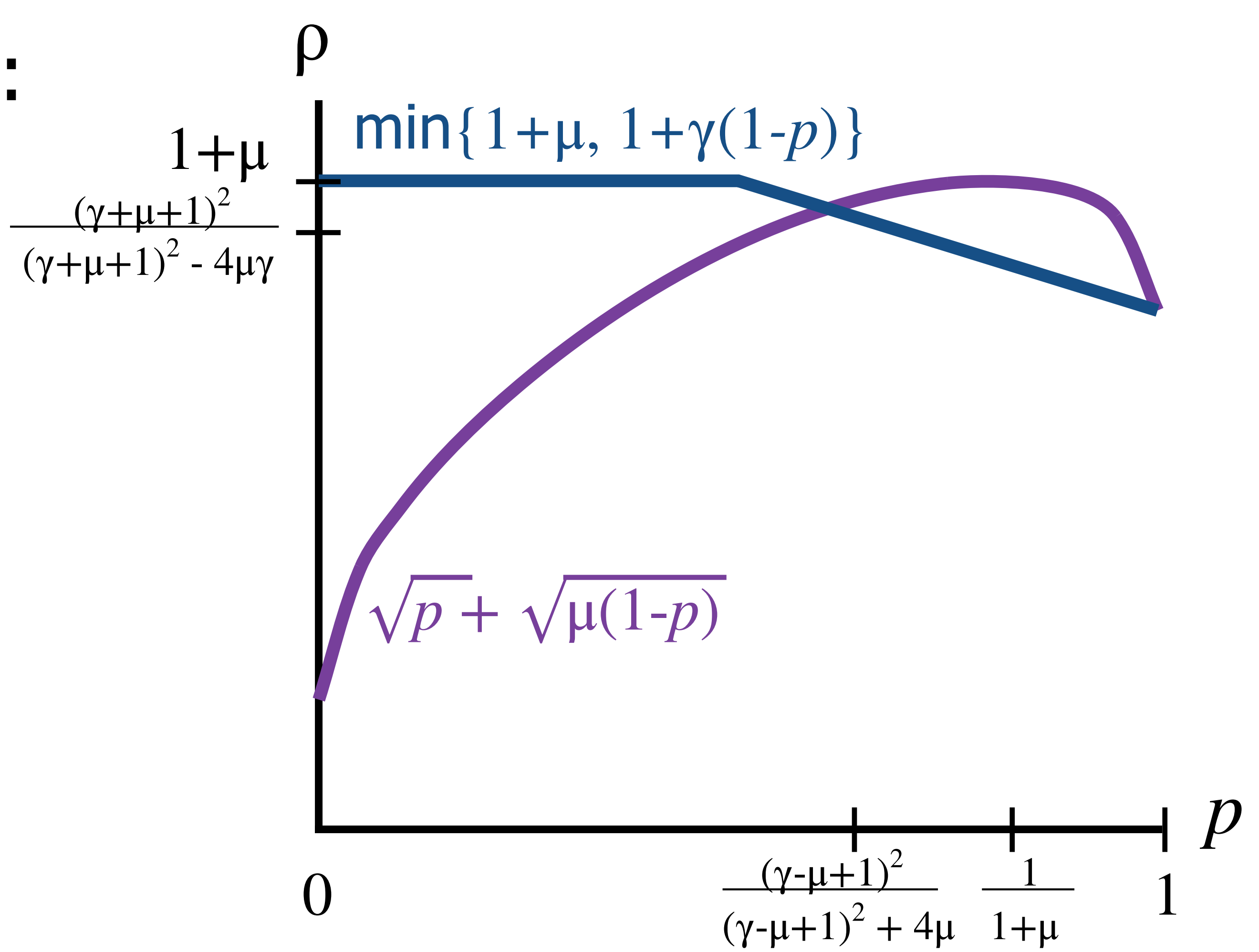

Approximation guarantee

Lemma

Approximation factor of Marcotte's Algorithm $\leq 1+\mu$.

- Let $x_e = f_e^*/z_e^*$
- by local optimality of z_e^*, we have $k_e = c_e'(x_e)x_e^2$

$$C(\mathbf{f}^*, \mathbf{z}^*) = \sum_{e \in E} c_e(f_e^*/z_e^*)f_e^* + k_e z_e^* = \sum_{e \in E} \Big(c_e(x_e) + c_e'(x_e)x_e\Big) f_e^*$$

- $\mu = \max_{\gamma} \Big(\gamma\big(1 - \frac{c_e(\gamma x)}{c_e(x)}\big)\Big) = \gamma \frac{c_e'(x)x}{c_e(x) + c_e'(x)x}$ s.t. $c_e(x/\gamma) = c_e(x) + c_e'(x)x$

$$\begin{aligned} C(\mathbf{f}^*, \mathbf{z}) &= \sum_{e \in E} c_e(x_e/\gamma_e)f_e^* + k_e \gamma_e z_e^* \\ &= \sum_{e \in E} \Big(c_e(x_e) + c_e'(x_e)x_e\Big) f_e^* + \gamma_e c_e'(x_e)x_e f_e^* \\ &\leq (1+\mu)C(\mathbf{f}^*, \mathbf{z}^*) \end{aligned}$$

Conclusion

- Many networks used by selfish users
- Network design with equilibrium constraints harder than regular network design
- Approximation algorithms based on relaxation
  - 49/41-approximation for affine costs
  - 8/5-approximation for general costs

- Thank you