Tree Path Labeling of Path Hypergraphs

Generalization of the Consecutive-ones Property

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An Illustration

- To introduce the combinatorial problem of TPL.

Study Group Accommodation problem

- A set of n **students** arrive for a summer course, say $\{a, b, c, d, e, f, g, h, i, j, k\}$, $n = 11$.

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- There are n single occupancy **apartments** in the university campus for their accommodation.
- All these apartments are placed such that streets connecting them do not form loops - **streets form a tree**

Study Group Accommodation problem

The problem

How should the students be allocated apartments such that:

- students of each study group are neighbours?

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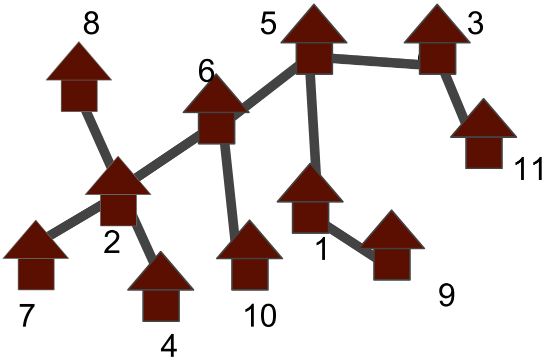
- i.e. a study group forms **a path in the tree**.

Study Group Accommodation problem



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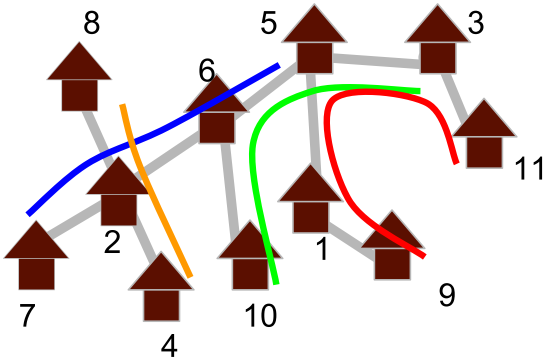

$R = \{g, h, i, j, k\}$
$\rightarrow \{9, 1, 5, 3, 11\}$
$B = \{a, b, e, g\}$
$\rightarrow \{7, 2, 6, 5\}$
$O = \{c, b, d\}$
$\rightarrow \{4, 2, 8\}$
$G = \{e, f, g, i\}$
$\rightarrow \{10, 6, 5, 3\}$

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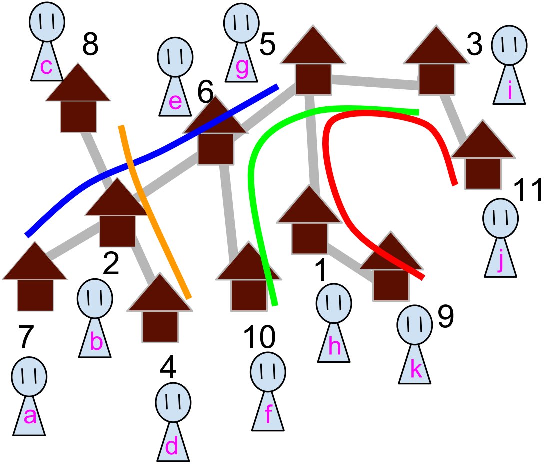

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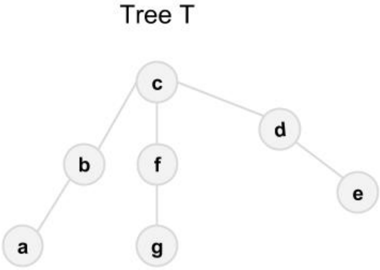
Path Labelling Example



F = { {3,4,6}, { 2,3,4,5}, {3,4,6,7}, {1,2,3}, {3,4,5,6,7} }
P = { {d,c,f}, {e,d,c,b}, { d,c,f,g}, {c,b,a}, {e,d,c,f,g} }

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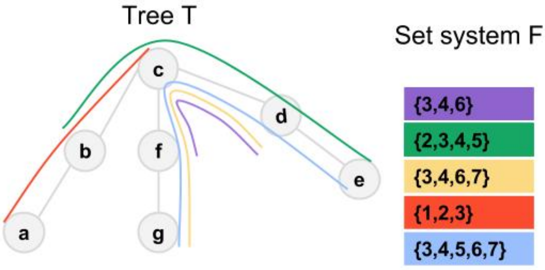

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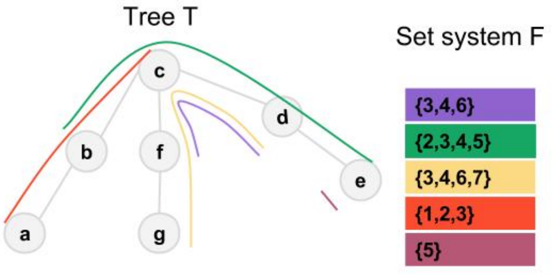

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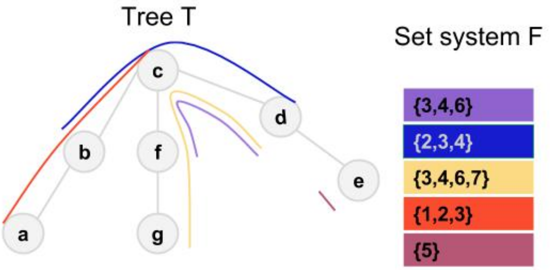

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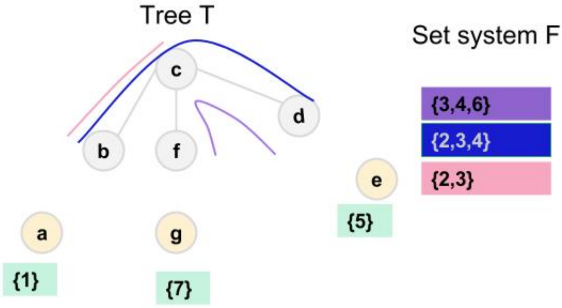

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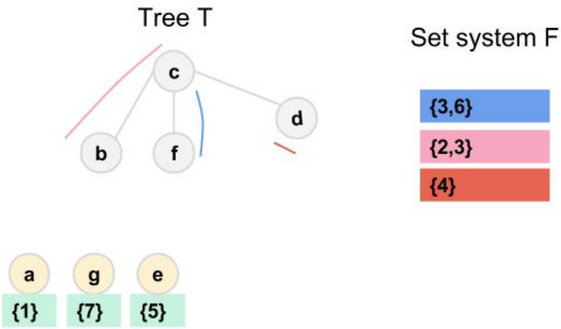

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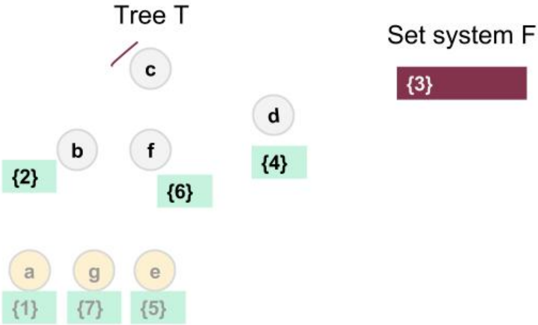

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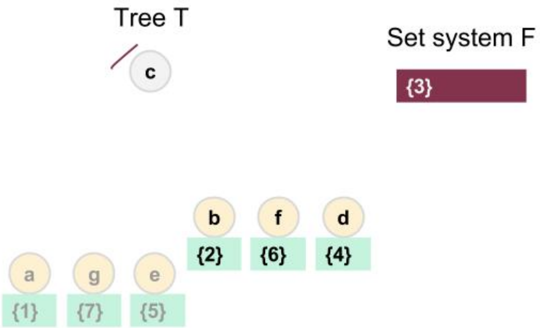

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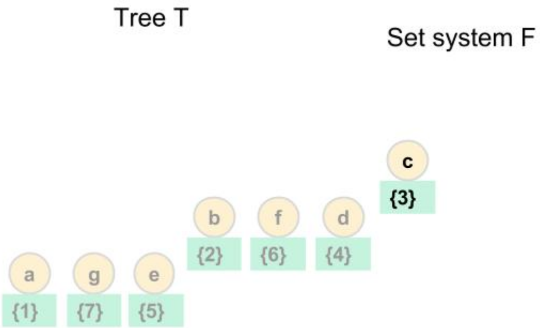

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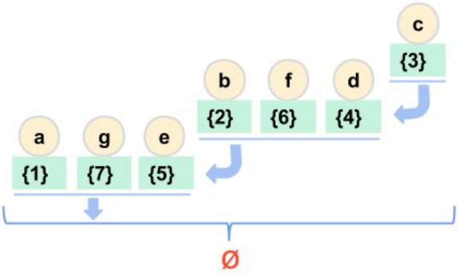

Tree Path Labeling of Path Hypergraphs

- The set of study groups → **Set system / Hypergraph**

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- The path mapping to study groups → **Tree Path Labeling (TPL)**
- The apartment allocation → **Path Hypergraph Isomorphism**

Tree Path Labeling of Path Hypergraphs

There *exists* an apartment allocation that "fits" the path mapping

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There *exists* an apartment allocation that gives some study group path mapping

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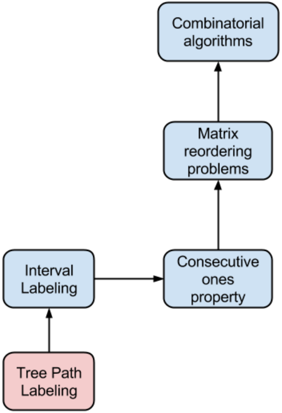
Tree Path Labeling of Path Hypergraphs

There *exists* a hypergraph isomorphism that "fits" the TPL
⇒ the TPL is FEASIBLE
There *exists* a hypergraph isomorphism that gives at least one feasible TPL
⇒ the hypergraph is a PATH HYPERGRAPH

Consecutive Ones → Path Labeling

The motivation



Feasible Tree Path Labeling

Input	A hypergraph $\mathcal{F}$ with vertex set U, a tree T, a set of paths $\mathcal{P}$ from T and a bijection $\ell : \mathcal{F} \to \mathcal{P}$.
Question	Does there exist a bijection $\phi : U \to V(T)$ such that ϕ when applied on any hyperedge in $\mathcal{F}$ will give the path mapped to it by the given tree path labeling ℓ. i.e., $\ell(S) = \{\phi(x) \mid x \in S\}$, for every hyperedge $S \in \mathcal{F}$.

Feasible Tree Path Labeling

- Is the given TPL ℓ of hypergraph $\mathcal{F}$ on tree T feasible?
- What is the hypergraph isomorphism $\phi : U \rightarrow V(T)$?
- Solvable in polynomial time.

Compute Feasible Tree Path Labeling

`Input`	A hypergraph $\mathcal{F}$ with vertex set U and a tree T.
`Question`	Does there exist a set of paths $\mathcal{P}$ from T and a bijection $\ell : \mathcal{F} \rightarrow \mathcal{P}$, such that Feasible Tree Path Labeling returns **true** on $(\mathcal{F}, T, \ell)$.

Compute Feasible Tree Path Labeling

- Is the given hypergraph $\mathcal{F}$ a path hypergraph w.r.t. target tree T?
- i.e. find at least one feasible tree path labeling $\ell : \mathcal{F} \to P$, P is a set of paths on T.
- Complexity is inconclusive for arbitrary trees, polynomial time for certain classes of trees.

Compute Interval Labeling

Compute Feasible Tree Path Labeling when target tree is an interval or path P_n

Compute Interval Labeling

- Is the given hypergraph $\mathcal{F}$ an interval hypergraph?
- Equivalent to consecutive ones property checking
- Finding an interval labelling
- Solvable in polynomial time.

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A characterization of feasible TPL

Intersection Cardinality Preserving Path Labeling (ICPPL)

A path labeling $(\mathcal{F}, \ell)$ on the given tree T s.t.

$$|S_1 \cap S_2 \cap S_3| = |\ell(S_1) \cap \ell(S_2) \cap \ell(S_3)|$$

for all not necessarily distinct $S_1, S_2, S_3 \in \mathcal{F}$

If T is a path, ICPPL is referred to as an Intersection Cardinality Preserving Interval Assignment (ICPIA)

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Theorem

A path labeling $(\mathcal{F}, \ell)$ on tree T is feasible iff it is an ICPPL.

Finding path hypergraph isomorphism ϕ

Given an ICPPL $(\mathcal{F}, \ell)$ on tree T

- Uses two filters to refine $(\mathcal{F}, \ell)$

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- When T is a path, problem becomes interval assignment. Use ICPIA

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- Repeat till no two paths share a leaf.

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`filter common leaf` $(\mathcal{F}, \ell)$

Lemma

Let $(\mathcal{F}', \ell')$ be the resulting labeling after applying `filter common leaf` *to TPL $(\mathcal{F}, \ell)$. If $(\mathcal{F}, \ell)$ is an ICPPL, $(\mathcal{F}', \ell')$ is also an ICPPL.*

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Proof.

- Induction on iteration of the filter.
- Invariants: $\ell_j(S)$ is a path, ℓ_j maintains ICPPL's intersection cardinality equations.
- ICPPL also preserves 4-way intersection cardinalities.

□

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- Remove S, P from $(\mathcal{F}, \ell)$

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- Repeat till there are no more unique paths for leaves. Call `filter common leaf` .

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- End if T is empty

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Critical part is finding $x \in S \setminus \cup_{S_i \neq S} S_i$

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Lemma

If $\ell(S)$ uniquely has a leaf, S_{priv} is non-empty where $S_{priv} = S \setminus \cup_{S_i \neq S} S_i$.

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Lemma

If $\ell(S)$ uniquely has a leaf, S_{priv} is non-empty where $S_{priv} = S \setminus \cup_{S_i \neq S} S_i$.

Proof.

- Let $\mathcal{F}' = S \cap S_i$ and $\ell'(S \cap S_i) = P \cap P_i$ for all $S_i \in \mathcal{F}$, $\ell(S_i) = P_i$.
- $S_{two} = supp(\mathcal{F}'), P_{two} = supp(\ell')$
- $(\mathcal{F}', \ell')$ is an ICPIA. Therefore $|S_{two}| = |P_{two}|$. Hence $|S_{priv}| = |P_{priv}|$. We know P has at least a leaf.

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Special case

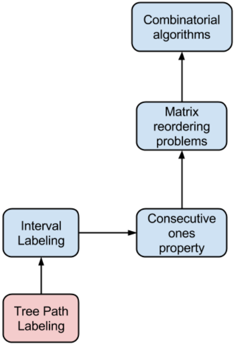
Interval assignment problem / COP

1. T is a path $\Longrightarrow$ paths in T are intervals
2. Only pairwise intersection cardinality needs to be preserved $\Longrightarrow$ ICPIA
3. Higher level intersection cardinalities preserved by **Helly Property**

This problem is equivalent to Consecutive Ones Property of binary matrices

Path Labeling → Graph Isomorphism

Application



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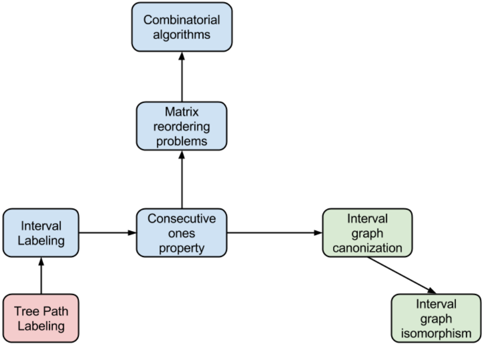

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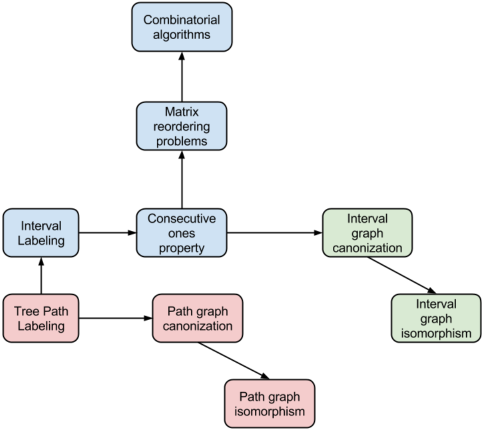

Thank you

Questions?