Cumulative Vehicle Routing Problem: A Column Generation Approach

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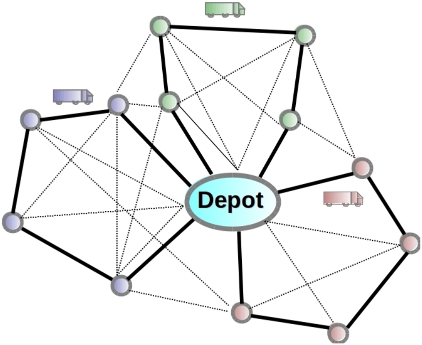
Outline

- Problem Definition
- Mathematical Formulation
- Column Generation Algorithm
- Simulation Results
- Worst Case Integrality Gap
- Conclusion

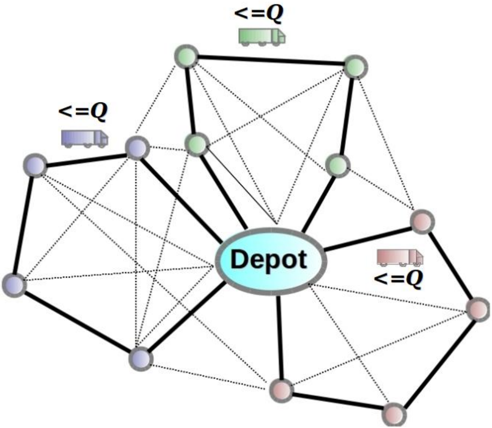
Vehicle Routing Problems (VRPs)

- Introduced by Dantzig and Ramser in 1959.
- **Input**: A fleet of delivery vehicles at the depot and customers with some demand.
- **Objective**: Scheduling the vehicles to meet the demand of the customers so as to minimize the total distance (or time).



Capacitated VRPs (CVRPs)

- **Capacitated VRPs (CVRPs)**: Variants of the VRPs with capacity constraint on the vehicles such that the load on each vehicle should not exceed the given capacity.



Fuel Cost (Expense)

- Major factor of the transportation cost.
- As much as 60% of the operational cost [Sahin et al., 2009]
    - Cargo in sea : 32%
    - Railroad : 46%
    - Road transportation : 60%
- Minimize fuel cost $\longrightarrow$ Reduced transportation cost

Figure : Source: Google Search

Fuel consumption

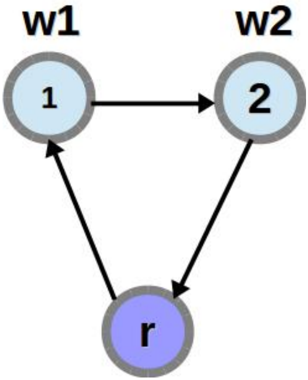
- Affected by various factors [Demir et al., 2011; Newman et al., 1989].
  - Distance traveled
  - Weight of the vehicle
  - Vehicle speed
  - Road inclination
  - Aerodynamic drag
  - Traffic congestion etc.

A Simplified model of fuel consumption

- Assumes that the fuel consumed per unit distance is proportional to the total weight of vehicle.
- Total weight of vehicle $=$ weight of the empty vehicle $+$ weight of cargo on the vehicle.
- Studied as
  - **Energy Minimizing in VRPs** by Kara et al., 2007.
  - **Cumulative VRPs** by Kara et al., 2008.
  - **Linear Model of Fuel Consumption for the Capacitated Vehicles** by Xiao et al., 2012.

Cumulative Cost Function

- a : cost of moving empty vehicle per unit distance
- b : cost of moving unit weight cargo per unit distance



$$a \cdot d(r, 1, 2, r) + b \cdot [w1 \cdot (d(1, 2, r)) + w2 \cdot d(2, r)]$$

Cumulative Vehicle Routing Problems (Cum-VRPs)

Input:

- A complete, weighted, undirected graph $G(V, E)$ (satisfying triangle inequality).
- A special depot node "r" and a set of customers located at other nodes.
- An object of positive weight d_i located at the customer node $i,\ i \in (V \setminus r)$.
- An empty truck located at the depot.
- The truck at any point of time can carry objects of total weight not exceeding Q.

Objective: Devise a travel schedule for the truck so that all the objects are brought to the depot and the **cumulative cost is minimized**. We allow the vehicle to **offload** cargo at the depot **an arbitrary number of times**.

Why the problem is interesting?

Generalizes two well known problems:

- Capacitated Vehicle Routing Problems (CVRPs)
- Capacitated Minimum Latency Problems (CMLPs)

Related Works

- Blum et al. (1994) gave a constant factor approximation algorithm (a single vehicle with infinite capacity, no intermediate offloading allowed).
  Travel schedule : a single TSP tour of the graph.
- Defined and Formulated by Kara et al. (2007,2008).
- Gaur et al. (2013) gave constant factor approximation algorithms for four variations of the problem.

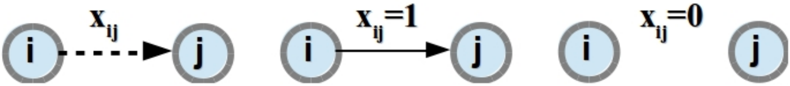
MILP Formulation

Some Notations

- n : number of customers.
- Q : capacity of the vehicle.
- a : cost of moving empty vehicle per unit distance.
- b : cost of moving unit weight cargo per unit distance.
- d_i : demand at customer node i. (integral)
- c_{ij} : distance between customer i and customer j.

Two decision Variables:

- $x_{ij} = 1$, if the vehicle visits customer j just after visiting customer i, otherwise $x_{ij} = 0$.



- y_{ij} : the weight of the cargo carried by the vehicle from customer i to customer j.

Objective Function

Cost of moving empty vehicle on solution tour

Cost of moving cargo from customer nodes to depot

$$min: \quad a \cdot \sum_{i=0}^{n} \sum_{j=0}^{n} (x_{ij} c_{ij}) + b \cdot \sum_{i=0}^{n} \sum_{j=0}^{n} (y_{ij} c_{ij})$$

a - cost **b - cost**

Objective Function is same as defined by Gaur et al. (2013).

Single-Visit Constraint

$$min: \quad \sum_{i=0}^{n}\sum_{j=0}^{n}((a \cdot x_{ij} + b \cdot y_{ij})c_{ij}) \qquad (1)$$

$$s.t.: \quad \sum_{i=0}^{n} x_{ij} = 1 \qquad (j = 1, 2, \cdots, n) \qquad (2)$$

Every customer node j will have exactly one incoming edge

Single-Visit Constraint

$$min: \quad \sum_{i=0}^{n}\sum_{j=0}^{n}((a \cdot x_{ij} + b \cdot y_{ij})c_{ij}) \qquad (1)$$

$$s.t.: \quad \sum_{i=0}^{n} x_{ij} = 1 \qquad (j = 1, 2, \cdots, n) \qquad (2)$$

$$\sum_{i=0}^{n} x_{ip} - \sum_{j=0}^{n} x_{pj} = 0 \qquad (p = 1, 2, \cdots, n) \qquad (3)$$

In-degree and out-degree is same at each customer node p

Flow Constraint

$$min: \quad \sum_{i=0}^{n}\sum_{j=0}^{n}((a \cdot x_{ij} + b \cdot y_{ij})c_{ij}) \qquad (1)$$

$$s.t.: \quad \sum_{i=0}^{n} x_{ij} = 1 \qquad (j = 1, 2, \cdots, n) \qquad (2)$$

$$\sum_{i=0}^{n} x_{ip} - \sum_{j=0}^{n} x_{pj} = 0 \qquad (p = 1, 2, \cdots, n) \qquad (3)$$

$$\sum_{j=0}^{n} y_{pj} - \sum_{i=0}^{n} y_{ip} = d_p \qquad (p = 1, 2, \cdots, n) \qquad (4)$$

Flow constraint : ensures that the supply at each customer node is picked

Capacity Constraint

$$min: \quad \sum_{i=0}^{n}\sum_{j=0}^{n}((a \cdot x_{ij} + b \cdot y_{ij})c_{ij}) \qquad (1)$$

$$s.t.: \quad \sum_{i=0}^{n} x_{ij} = 1 \qquad (j = 1, 2, \cdots, n) \qquad (2)$$

$$\sum_{i=0}^{n} x_{ip} - \sum_{j=0}^{n} x_{pj} = 0 \qquad (p = 1, 2, \cdots, n) \qquad (3)$$

$$\sum_{j=0}^{n} y_{pj} - \sum_{i=0}^{n} y_{ip} = d_p \qquad (p = 1, 2, \cdots, n) \qquad (4)$$

$$y_{ij} \leq Q \cdot x_{ij} \qquad (i, j = 1, 2, \cdots, n) \qquad (5)$$

capacity constraint

MILP Formulation due to Kara et al. [3, 4]

$$min: \quad \sum_{i=0}^{n}\sum_{j=0}^{n}((a \cdot x_{ij} + b \cdot y_{ij})c_{ij}) \qquad (1)$$

$$s.t.: \quad \sum_{i=0}^{n} x_{ij} = 1 \qquad (j = 1, 2, \cdots, n) \qquad (2)$$

$$\sum_{i=0}^{n} x_{ip} - \sum_{j=0}^{n} x_{pj} = 0 \qquad (p = 1, 2, \cdots, n) \qquad (3)$$

$$\sum_{j=0}^{n} y_{pj} - \sum_{i=0}^{n} y_{ip} = d_p \qquad (p = 1, 2, \cdots, n) \qquad (4)$$

$$y_{ij} \leq Q \cdot x_{ij} \qquad (i, j = 1, 2, \cdots, n) \qquad (5)$$

$$x_{ij} \in \{0, 1\} \qquad (i, j = 0, 1, 2, \cdots, n) \qquad (6)$$

$$y_{ij} \geq 0 \qquad (i, j = 0, 1, 2, \cdots, n) \qquad (7)$$

An Equivalent Set cover formulation

Given by Balinski and Quandt (1964) for VRP.

$$\text{min}: \quad \sum_{j \in R} \theta_j \cdot \alpha_j \tag{8}$$

$$\text{s.t.}: \quad \sum_{j \in R} z_{ij} \cdot \alpha_j \geq 1 \qquad (i = 1, 2, \cdots, n) \tag{9}$$

$$\alpha_j \in \{0, 1\}. \tag{10}$$

Set cover formulation

Feasible subset of customer nodes

$$\min: \sum_{j \in R} \theta_j \cdot \alpha_j \tag{8}$$

$$s.t.: \sum_{j \in R} z_{ij} \cdot \alpha_j \geq 1 \qquad (i = 1, 2, \cdots, n) \tag{9}$$

$$\alpha_j \in \{0, 1\}. \tag{10}$$

Set cover formulation

Feasible subset of customer nodes

Set of all feasible subsets j

$$\min: \sum_{j \in R} \theta_j \cdot \alpha_j \tag{8}$$

$$s.t.: \sum_{j \in R} z_{ij} \cdot \alpha_j \geq 1 \qquad (i = 1, 2, \cdots, n) \tag{9}$$

$$\alpha_j \in \{0, 1\}. \tag{10}$$

Set cover formulation

Feasible subset of customer nodes

Minimum cumulative cost for serving the demand of the customers in subset j from the depot

Set of all feasible subsets j

$$min: \sum_{j \in R} \theta_j \cdot \alpha_j \quad (8)$$

$$s.t.: \sum_{j \in R} z_{ij} \cdot \alpha_j \geq 1 \qquad (i = 1, 2, \cdots, n) \quad (9)$$

$$\alpha_j \in \{0, 1\}. \quad (10)$$

Set cover formulation

Feasible subset of customer nodes

Minimum cumulative cost for serving the demand of the customers in subset j from the depot

Set of all feasible subsets j

whether or not customer node i is in subset j

$$min: \sum_{j \in R} \theta_j \cdot \alpha_j \tag{8}$$

$$s.t.: \sum_{j \in R} z_{ij} \cdot \alpha_j \geq 1 \qquad (i = 1, 2, \cdots, n) \tag{9}$$

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Set cover formulation

Feasible subset of customer nodes

Minimum cumulative cost for serving the demand of the customers in subset j from the depot

$$min: \sum_{j \in R} \theta_j \cdot \alpha_j \quad (8)$$

Set of all feasible subsets j

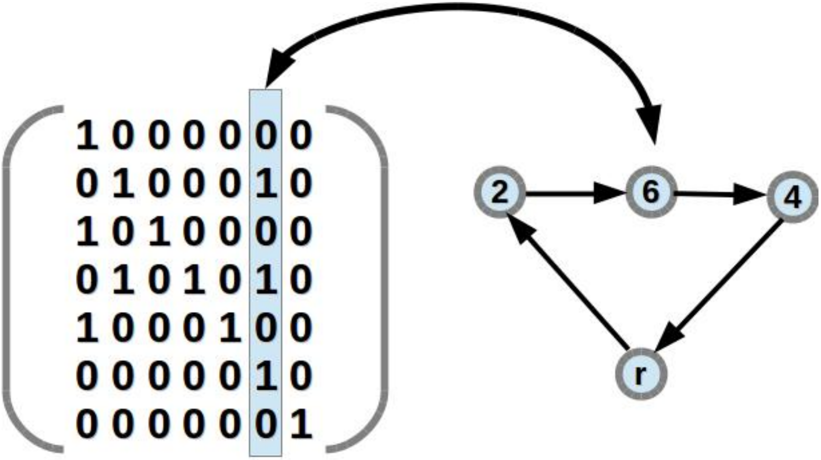
$$s.t.: \sum_{j \in R} z_{ij} \cdot \alpha_j \geq 1 \qquad (i = 1, 2, \cdots, n) \quad (9)$$

$$\alpha_j \in \{0, 1\}. \quad (10)$$

whether or not subset j is in the optimal solution

whether or not customer node i is in subset j

Coefficient Matrix (Z)



Column

Feasible Cycle / Subset

Set cover formulation : Two problems

Feasible subset of customer nodes

Minimum cumulative cost for serving the demand of the customers in subset j from the depot

Set of all feasible subsets j

$$\min : \sum_{j \in R} \theta_j \cdot \alpha_j \quad (8)$$

$$s.t. : \sum_{j \in R} z_{ij} \cdot \alpha_j \geq 1 \qquad (i = 1, 2, \cdots, n) \quad (9)$$

$$\alpha_j \in \{0, 1\}. \quad (10)$$

whether or not subset j is in the optimal solution

whether or not customer node i is in subset j

Solution : Restricted Master Problem

$$\begin{aligned} min : \quad & \sum_{j \in R'} \theta_j \cdot \alpha_j \\ s.t. : \quad & \sum_{j \in R'} z_{ij} \cdot \alpha_j \geq 1 \qquad (i = 1, 2, \cdots, n) \\ & \alpha_j \in \{0, 1\}. \end{aligned}$$

Restricted Master Problem (RMP)

It can be easily computed for all *j* in *R'*

$$\min: \quad \sum_{j \in R'} \theta_j \cdot \alpha_j$$

A very small subset of R

$$s.t.: \quad \sum_{j \in R'} z_{ij} \cdot \alpha_j \geq 1 \qquad (i = 1, 2, \cdots, n)$$

$$\alpha_j \in \{0, 1\}.$$

Pricing Sub-Problem: ERCSPP

$$min: \quad \sum_{i=0}^{n}\sum_{j=0}^{n}((a \cdot x_{ij} + b \cdot y_{ij})c_{ij}) - \sum_{i=1}^{n}(\pi_i \cdot \sum_{j=0}^{n} x_{ji}) \tag{11}$$

$$s.t.: \quad \sum_{j=1}^{n} x_{0j} = 1 \tag{12}$$

$$\sum_{j=1}^{n} x_{j0} = 1 \tag{13}$$

$$\sum_{i=0}^{n} x_{ip} - \sum_{j=0}^{n} x_{pj} = 0 \qquad (p = 1, 2, \cdots, n) \tag{14}$$

$$\sum_{j=0}^{n} y_{pj} - \sum_{i=0}^{n} y_{ip} = d_p \cdot \sum_{k=0}^{n} x_{kp} \qquad (p = 1, 2, \cdots, n) \tag{15}$$

$$y_{ij} \leq Q \cdot x_{ij} \qquad (i, j = 1, 2, \cdots, n) \tag{16}$$

$$x_{ij} \in \{0, 1\} \qquad (i, j = 0, 1, 2, \cdots, n) \tag{17}$$

$$y_{ij} \geq 0 \qquad (i, j = 0, 1, 2, \cdots, n) \tag{18}$$

Pricing Sub-Problem: ERCSPP

Objective function: reduced cost of a cycle

$$\min: \overbrace{\sum_{i=0}^{n}\sum_{j=0}^{n}((a \cdot x_{ij} + b \cdot y_{ij})c_{ij}) - \sum_{i=1}^{n}(\pi_i \cdot \sum_{j=0}^{n} x_{ji})} \quad (11)$$

$$\text{s.t.}: \left.\begin{aligned} &\sum_{j=1}^{n} x_{0j} = 1 \quad (12) \\ &\sum_{j=1}^{n} x_{j0} = 1 \quad (13) \end{aligned}\right\}$$

$$\sum_{i=0}^{n} x_{ip} - \sum_{j=0}^{n} x_{pj} = 0 \qquad (p = 1, 2, \cdots, n) \quad (14)$$

$$\sum_{j=0}^{n} y_{pj} - \sum_{i=0}^{n} y_{ip} = d_p \cdot \sum_{k=0}^{n} x_{kp} \qquad (p = 1, 2, \cdots, n) \quad (15)$$

$$y_{ij} \leq Q \cdot x_{ij} \qquad (i, j = 1, 2, \cdots, n) \quad (16)$$

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Pricing Sub-Problem: ERCSPP

Objective function: reduced cost of a cycle

$$min: \overbrace{\sum_{i=0}^{n}\sum_{j=0}^{n}((a \cdot x_{ij} + b \cdot y_{ij})c_{ij}) - \sum_{i=1}^{n}(\pi_i \cdot \sum_{j=0}^{n} x_{ji})} \quad (11)$$

Single cycle: one outgoing edge from the depot, and one incoming edge into the depot.

$$s.t.: \sum_{j=1}^{n} x_{0j} = 1 \quad (12)$$

$$\sum_{j=1}^{n} x_{j0} = 1 \quad (13)$$

$$\sum_{i=0}^{n} x_{ip} - \sum_{j=0}^{n} x_{pj} = 0 \qquad (p = 1, 2, \cdots, n) \quad (14)$$

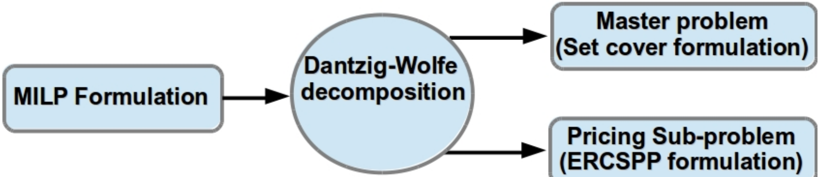
$$\sum_{j=0}^{n} y_{pj} - \sum_{i=0}^{n} y_{ip} = d_p \cdot \sum_{k=0}^{n} x_{kp} \qquad (p = 1, 2, \cdots, n) \quad (15)$$

$$y_{ij} \leq Q \cdot x_{ij} \qquad (i, j = 1, 2, \cdots, n) \quad (16)$$

$$x_{ij} \in \{0, 1\} \qquad (i, j = 0, 1, 2, \cdots, n) \quad (17)$$

$$y_{ij} \geq 0 \qquad (i, j = 0, 1, 2, \cdots, n) \quad (18)$$

Dantzig-Wolfe decomposition (1960)



$$
\begin{aligned}
min: & \sum_{j \in R'} \theta_j \cdot \alpha_j \\
s.t.: & \sum_{j \in R'} z_{ij} \cdot \alpha_j \geq 1 \qquad (i = 1, 2, \cdots, n) \\
& \alpha_j \in \{0, 1\}.
\end{aligned}
$$

Restricted Master Problem: Set cover formulation

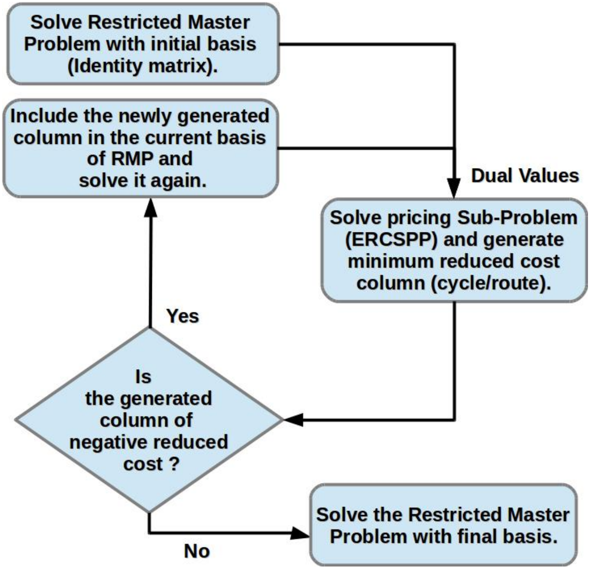
$$
\begin{aligned}
min: & \sum_{i=0}^{n} \sum_{j=0}^{n} ((a \cdot x_{ij} + b \cdot y_{ij}) c_{ij}) - \sum_{i=1}^{n} (\pi_i \cdot \sum_{j=0}^{n} x_{ji}) \\
s.t.: & \sum_{j=1}^{n} x_{0j} = 1 \\
& \sum_{j=1}^{n} x_{j0} = 1 \\
& \sum_{i=0}^{n} x_{ip} - \sum_{j=0}^{n} x_{pj} = 0 \qquad (p = 1, 2, \cdots, n) \\
& \sum_{j=0}^{n} y_{pj} - \sum_{i=0}^{n} y_{ip} = d_p \cdot \sum_{k=0}^{n} x_{kp} \qquad (p = 1, 2, \cdots, n) \\
& y_{ij} \leq Q \cdot x_{ij} \qquad (i, j = 1, 2, \cdots, n) \\
& x_{ij} \in \{0, 1\} \qquad (i, j = 0, 1, 2, \cdots, n) \\
& y_{ij} \geq 0 \qquad (i, j = 0, 1, 2, \cdots, n)
\end{aligned}
$$

Pricing Sub-Problem: ERCSPP

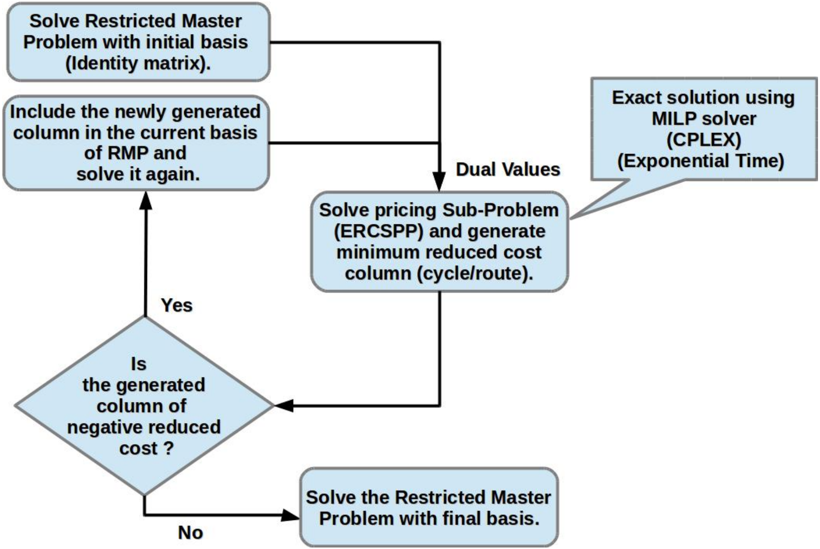
Algorithms

- Column Generation: Produce fractional solution.
- Dynamic Programming : Solve pricing sub-problem
- Randomized Rounding: Generating integral solution from fractional solution.

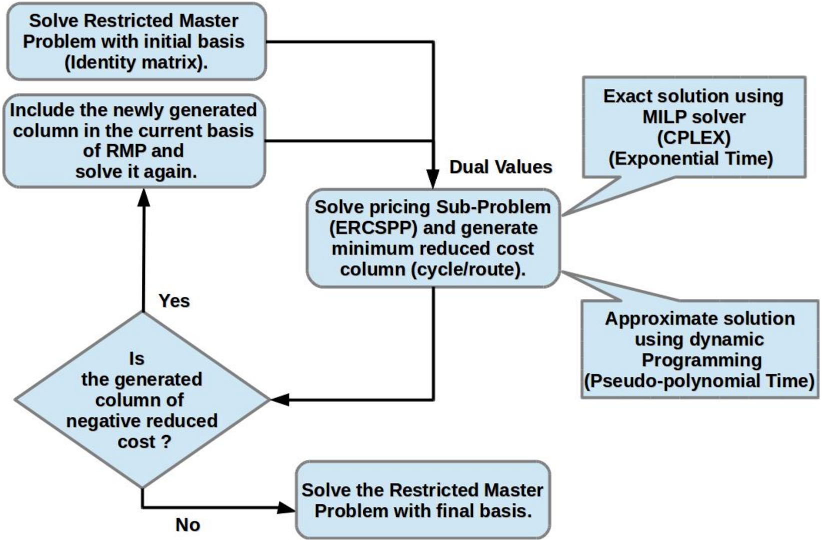
Column Generation



Column Generation



Column Generation



Simulation Results

- Three different cases for values of (a, b):
  - $a = 1, b = 0$
  - $a = 0, b = 1$
  - $a = 1, b = Q$
- Instances : [A-set, B-set, P-set, E-set, and RY-instance][1].
- MILP formulation for the sub-problem was also solved using CPLEX MILP solver.
- Average cost of the integral solution obtained over 20 rounding.
- Time-out : 3 Hours.

[1] *http://neo.lcc.uma.es/vrp/vrp-instances/capacitated-vrp-instances*

Table : CVRP's ($a = 1, b = 0$) and instances from E-set

Instances	Q	CVRP			MILP					DP			Avg	Avg Fact
		OPT	LB	NOC	TT	DT	SPT	LB	NOC	TT	DT	SPT	R-value	
E-n7	3		100.00	3	0.9	0.3	0.6	100.00	3	0.3	0.3	0.0	116.20	
E-n13-k4	6000	247	247.00	23	10.3	0.5	9.8	247.00	23	60.5	0.4	60.0	247.00	1.00
E-n22-k4	6000	375	373.71	37	48.4	0.7	47.7	373.71	40	634.0	0.7	633.3	412.75	1.10
E-n23-k3	4500	569	558.95	62	340.0	1.3	338.6	558.95	73	1034.8	1.6	1033.2	749.80	1.32
E-n30-k(3, 4)	4500	534	484.10	112	6954.1	3.0	6951.1	484.10	103	3539.4	2.4	3536.9	692.15	1.30
E-n31-k7	140	379	309.00	43	65.7	0.8	64.8	309.00	49	46.8	0.9	45.8	643.80	1.70
E-n33-k4	8000	835												
E-n51-k5	160	521	517.06	284	8154.2	34.5	8119.5	517.08	309	1840.7	47.7	1792.9	685.60	1.32
E-n76-k7	220	682												
E-n76-k8	180	735												
E-n76-k10	140	830						812.45	397	7013.9	65.9	6947.9	1140.60	1.37
E-n76-k(14,15)	100	1021						1002.75	283	3408.8	15.9	3392.8	1412.55	1.38
E-n101-k8	200	815												
E-n101-k14	112	1071												

LB : Cost of the fractional solution to the master problem
Avg R-value : Average integral cost after rounding.
NOC : Number of columns (generated).
TT : Total time, **DT :** Dual time, **SPT :** Sub-problem time.

Theoretical bound of 2.5 by Altinkemer and Gavish (1990) [$2 + \epsilon$ in case of Euclidean CVRP].

Table : CMLP's ($a = 0, b = 1$) and instances from E-set

Instances	Q	MILP					DP					Avg
		LB	NOC	TT	DT	SPT	LB	NOC	TT	DT	SPT	R-value
E-n7	3	72.00	2	0.5	0.3	0.2	72.00	2	0.1	0.0	0.0	76.00
E-n13-k4	6000	429400.00	5	2.1	0.3	1.7	429400.00	5	15.1	0.1	15.1	429400.00
E-n22-k4	6000	628700.00	1	0.9	0.2	0.6	628700.00	1	30.9	0.0	30.9	628700.00
E-n23-k3	4500	407318.00	1	1.0	0.2	0.7	407318.00	1	28.3	0.2	28.0	407318.00
E-n30-k(3, 4)	4500	577025.00	3	2.7	0.3	2.4	577025.00	3	136.5	0.3	136.1	577025.00
E-n31-k7	140	10071.00	42	90.1	0.7	89.3	10071.00	44	41.8	0.5	41.2	14536.00
E-n33-k4	8000	2296050.00	8	12.6	0.3	12.2	2296050.00	11	960.9	0.4	960.5	2296050.00
E-n51-k5	160	18017.00	3	7.6	0.3	7.3	18017.00	3	23.5	0.3	23.2	18017.00
E-n76-k7	220	32010.00	3	43.0	0.3	42.7	32010.00	7	224.8	0.1	224.6	32010.00
E-n76-k8	180	32010.00	3	40.8	0.3	40.5	32010.00	7	183.1	0.3	182.7	32010.00
E-n76-k10	140	32010.00	4	40.3	0.3	40.0	32010.00	7	139.9	0.4	139.5	32010.00
E-n76-k(14,15)	100	32010.00	3	23.7	0.3	23.4	32010.00	7	96.6	0.3	96.2	32010.00
E-n101-k8	200						36614.00	20	1257.0	0.5	1256.5	37043.45
E-n101-k14	112						36614.00	20	690.2	0.2	690.0	36969.55

Most of the time : Objective function = sum of the shortest path from depot to each client

Table : Cu-VRP's ($a = Q, b = 1$) and instances from E-set

Instances	Q	MILP					DP					Avg
		LB	NOC	TT	DT	SPT	LB	NOC	TT	DT	SPT	R-value
E-n7	3	388.00	2	0.5	0.2	0.2	388.00	2	0.3	0.3	0.0	456.00
E-n13-k4	6000	2067300.00	19	8.0	0.4	7.5	2067300.00	19	50.5	0.4	50.0	2067300.00
E-n22-k4	6000	3123000.00	38	50.4	0.7	49.7	3129800.00	31	495.1	0.6	494.5	3336200.00
E-n23-k3	4500	3243577.00	51	1084.3	1.0	1083.3	3243577.00	26	377.3	0.5	376.7	3243577.00
E-n30-k(3, 4)	4500						2992922.06	50	1735.6	1.0	1734.5	3368842.50
E-n31-k7	140	58766.00	41	91.9	0.8	91.1	58766.00	44	42.1	0.8	41.3	147134.85
E-n33-k4	8000											
E-n51-k5	160						116705.95	267	1576.7	26.3	1550.3	155593.95
E-n76-k7	220											
E-n76-k8	180						182427.33	459	10617.0	124.0	10492.9	261691.30
E-n76-k10	140						160979.54	336	5917.6	32.2	5885.3	234395.35
E-n76-k(14,15)	100						141236.88	261	3142.7	10.8	3131.8	187975.15
E-n101-k8	200											
E-n101-k14	112											

Theoretical bound of 4 by Gaur et al. (2013) [$3.414 + \epsilon$ in case of Euclidean Cu-VRP].

Notations: Integrality Gap Analysis for Equal-weight case

- C^*: An optimal traveling salesperson tour.
- Q: Capacity of the vehicle.
- d_i: Distance between vertex i and the depot.
- X^*_{eq}: Cost of the optimal integral solution.
- X^{LP}_{eq}: Cost of the optimal fractional solution.
- X^{ITP}_{eq}: Fuel consumption on the solution tour from Gaur et al. (2013).
- Z^{LP}_{eq}: Cost of the optimal fractional solution to the corresponding CVRP instance.

Integrality Gap

$$\frac{X_{eq}^{*}}{X_{eq}^{LP}} \leq ?$$

Similar to the Bramel and Simchi-Levi's analysis for CVRP in the book by Toth and Vigo (2001)

Upper bound on X^*_{eq}

Theorem 1

[Gaur et al., 2013] *Let $\beta > 0$ be a positive rational number. Then, there exists a cluster partition $P = [1, i_1, i_2, \dots, i_{k-1}, n]$ using ITP of C^* with total fuel consumption*

$$X_{eq}^{ITP} \leq \left(1 + \frac{2}{\beta}\right) \cdot b \cdot \left(\sum_{i=1}^{n} d_i\right) + \left(1 + \frac{\beta}{2}\right) a|C^*| + 2a\frac{\sum_{i=1}^{n} d_i}{Q}. \quad (19)$$

It is trivial to note that $X^*_{eq} \leq X^{ITP}_{eq}$. So, we get:

$$X^*_{eq} \leq \left(1 + \frac{2}{\beta}\right) \cdot b \cdot \left(\sum_{i=1}^{n} d_i\right) + \left(1 + \frac{\beta}{2}\right) a|C^*| + 2a\frac{\sum_{i=1}^{n} d_i}{Q}. \quad (20)$$

Lower Bounds for X_{eq}^{LP}

$$X_{eq}^{LP} \geq a \cdot Z_{eq}^{LP} + b\left(\sum_{i=1}^{n} d_i\right). \tag{21}$$

Using $Z_{eq}^{LP} \geq \frac{2}{Q}\sum_{i=1}^{n} d_i$, due to the lower bound given by Haimovich and Rinnooy Kan (1985) for CVRP, we can rewrite:

$$X_{eq}^{LP} \geq a \cdot \frac{2}{Q}\sum_{i=1}^{n} d_i + b\left(\sum_{i=1}^{n} d_i\right). \tag{22}$$

Using $|C^*| \leq \frac{3}{2} Z_{eq}^{LP}$ due to Held and Karp (1970), we can get:

$$a \cdot |C^*| + \frac{3}{2} b\left(\sum_{i=1}^{n} d_i\right) \leq \frac{3}{2} X_{eq}^{LP}. \tag{23}$$

Worst Case Bound on Integrality Gap: Equal Weight Case

Now, we re-write the equation (20) as :

$$X^*_{eq} \leq \left[2a \cdot \frac{\sum_{i=1}^{n} d_i}{Q} + b \cdot \left(\sum_{i=1}^{n} d_i\right)\right] + max[\frac{2}{3} \cdot \frac{2}{\beta}, 1 + \frac{\beta}{2}] \cdot \left[a|C^*| + b \cdot \left(\sum_{i=1}^{n} d_i\right)\right]$$

or using equation (22) and equation (23), we can write:

$$X^*_{eq} \leq X^{LP}_{eq} + \frac{3}{2} \cdot max[\frac{4}{3\beta}, 1 + \frac{\beta}{2}] \cdot X^{LP}_{eq}$$

or

$$\frac{X^*_{eq}}{X^{LP}_{eq}} \leq 1 + \frac{3}{2} \cdot max[\frac{4}{3\beta}, 1 + \frac{\beta}{2}] \quad (24)$$

A minimum factor can be obtained for $\beta > 0$, when $\frac{4}{3\beta} = 1 + \frac{\beta}{2}$ or $\beta = \frac{\sqrt{33}}{3} - 1$, that results :

$$\frac{X^*_{eq}}{X^{LP}_{eq}} \leq 3.18614 \quad (25)$$

Worst Case Bound on Integrality Gap: Unequal Weight Case

Similar to the equation (24) for equal demands case, we can get following equation for unequal weights case:

$$\frac{X^*_{uneq}}{X^{LP}_{uneq}} \leq 2 + \frac{3}{2} \cdot max[\frac{3}{2} \cdot (\frac{2}{\beta} - 1), 1 + \frac{\beta}{2}] \qquad (26)$$

A minimum factor can be obtained for $\beta > 0$, when $\frac{3}{2} \cdot (\frac{2}{\beta} - 1) = 1 + \frac{\beta}{2}$ or $\beta = \frac{2}{3}$, that results :

$$\frac{X^*_{uneq}}{X^{LP}_{uneq}} \leq 4 \qquad (27)$$

Conclusion

- Empirically evaluation of the performance of column generation based approximation algorithm for the cumulative VRP.
- Solved a set cover type formulation for the cumulative VRP problem using column generation.
- Simulation results are better than the worst-case bounds on the approximation algorithms developed using the ITP technique due to Gaur et al. (2013).
- Scalability : branch cut and price based approach Vs our approach.

Future Work

- Theoretical bounds on the approximability of DP.
- Better than $O(\log n)$ factor analysis for the integral solution due to rounding.
- Other factors affecting the fuel consumption such as traffic congestion, road inclination, aerodynamic drag, engine characteristics of the vehicle etc. can be considered for complex modeling.
- The approximability of cumulative VRPs when the number of offloadings allowed is given as input, remains an open question.

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Thank You!

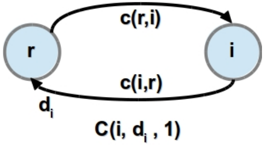

Any
Questions

Some Notations : Dynamic Program (DP)

Similar to the Dynamic Program due to Lysgaard and Wohlk (2013).

- *Elementary route* : cycle starts at the depot (r) and ends at the depot (r) without re-visiting any node.
- $C(i, q, x)$: the cost of the minimal cost route that collects q units of goods, visits a total of x clients and the last node visited before returning to the depot is client i.
- $R(i, q, x)$: the route that achieves this minimal cost. Note that there might be more than one route which attains the minimal cost.
- $c(r, i)$: the shortest distance between the depot r and node i.
- y_i : the dual value associated with client i.

Dynamic Programming for Pricing Sub-Problem

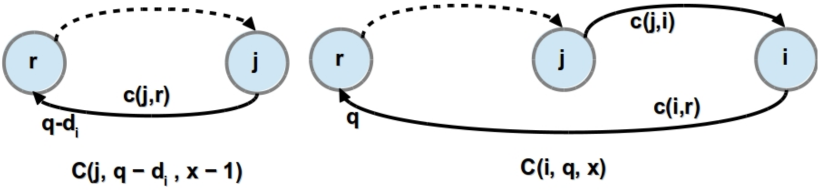


Initialization:

For $i \in [1..n]$, $q \in [d_1, d_2, \ldots, d_n]$, and $x = 1$

$$C(i, q, x) = a.c(r, i) + (a + b.d_i).c(i, r) - y_i$$

Dynamic Programming for Pricing Sub-Problem



Updation:

$$C(i,q,x) = \min_{j \neq i, i \notin R(j,q-d_i,x-1)} \left\{ \begin{array}{c} C(j,q-d_i,x-1) - (a+b(q-d_i))c(j,r) + \\ (a+b(q-d_i))c(j,i) + (a+b.q)c(i,r) - y_i \end{array} \right\}$$

$R(i,q,x)$ has to be updated accordingly.

Analysis of DP

- Time Complexity: $O(n^3 Q)$.
- Recurrence relation does not consider all the paths.
- Principle of optimality may not hold for the recurrence relation.
- We get a ***minimal*** reduced cost column : an approximate solution to the pricing sub-problem
- The paths returned by the dynamic programming algorithm are close to the optimal: evidence by simulations.

Rounding Algorithm

Algorithm 1 RA

(Input) Solution (x, A) from CG algorithm.
Start with an empty set of cycles S.
while S is not a feasible cover (every customer is in some cycle) **do**
    Round all the fractional solution x to 1 with probability $x(i)$.
    Add all the cycles (columns) with rounded $x(i) = 1$ to S.
end while
Take the cycles in S in a random order $\{S_1, S_2, \cdots, S_k\}$.
for $i = 1$ to k **do**
    if $S \setminus \{S_i\}$ is a feasible cover **then**
        Remove cycle S_i from S.
    end if
end for

Rounding Algorithm

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end for

Reverse delete step