A PTAS for the Metric Case of the Minimum Sum-Requirement Communication Spanning Tree Problem

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m-SROCT Definition

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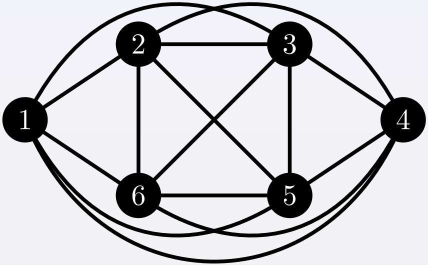
Metric Case of the Optimum Sum-Requirement Communication Spanning Tree Problem

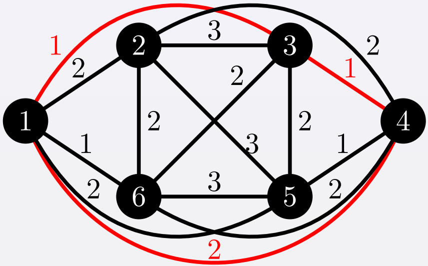
Input: a complete graph $G = \langle V, E \rangle$;



m-SROCT Definition

Metric Case of the Optimum Sum-Requirement Communication Spanning Tree Problem

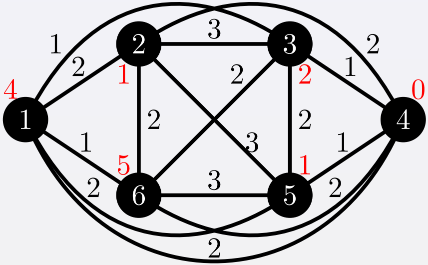
Input: a complete graph $G = \langle V, E \rangle$;
a metric length $\omega : E \to \mathbb{Q}_+$;



m-SROCT Definition

Metric Case of the Optimum Sum-Requirement Communication Spanning Tree Problem

Input: a complete graph $G = \langle V, E \rangle$;
a metric length $\omega : E \to \mathbb{Q}_+$;
a routing weight $r : V \to \mathbb{Q}_+$.

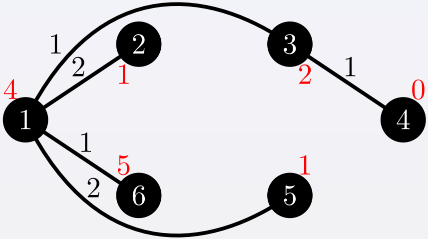


Time:

m-SROCT Definition

Metric Case of the Optimum Sum-Requirement Communication Spanning Tree Problem

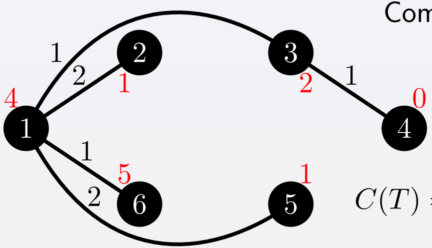
Output: to find a spanning tree $T \subseteq G$,



m-SROCT Definition

Metric Case of the Optimum Sum-Requirement Communication Spanning Tree Problem

Output: to find a spanning tree $T \subseteq G$, that minimizes the communication cost $C(T)$.



Communication requirement

$$\psi : V \times V \to \mathbb{Q}_+$$

$$\psi(u,v) = r(u) + r(v)$$

$$C(T) = \sum_{u,v \in V} \psi(u,v) d_T(u,v).$$

Results in Literature

Results in Literature

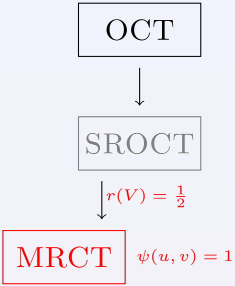
OCT — no routing weights; arbitrary non-negative $\psi(u, v)$

↓

SROCT

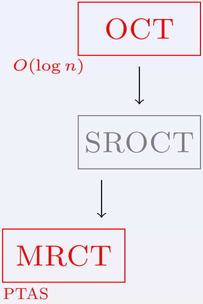
Hu (1974).

Results in Literature



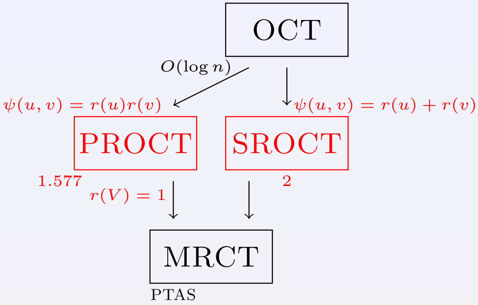
Johnson et al. (1978).

Results in Literature



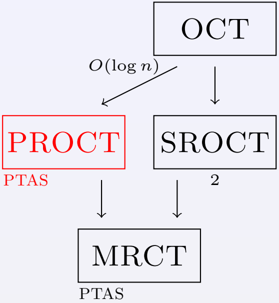
Wu et al. (1998).

Results in Literature



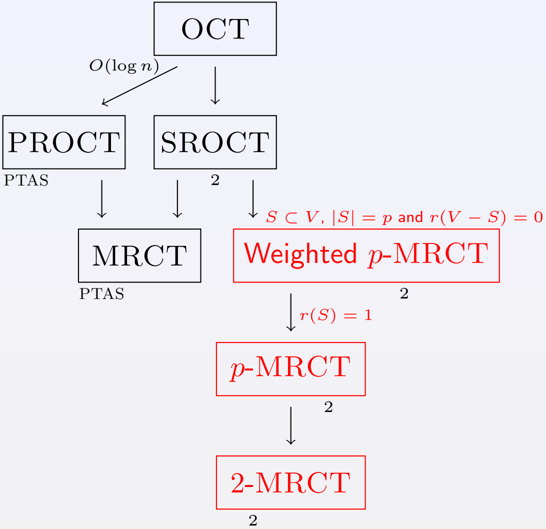
Wu et al. (2000).

Results in Literature



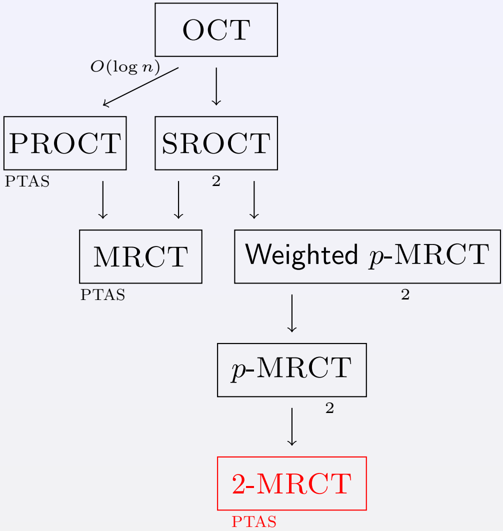
Wu et al. (2000).

Results in Literature



Farley et al. (2000).

Results in Literature



Wu (2002).

PTAS for m-SROCT

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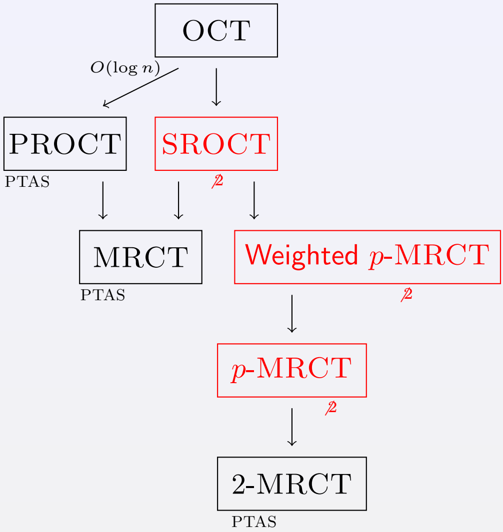


PTAS for m-SROCT

Our approach

- For a fixed integer k we show how to find an optimal k-star.
- We prove that for every $0 < \delta \leq \frac{1}{2}$, an optimal k-star, with k depending on δ, is a $\frac{1}{1-\delta}$-approximation of the optimal value for m-SROCT.

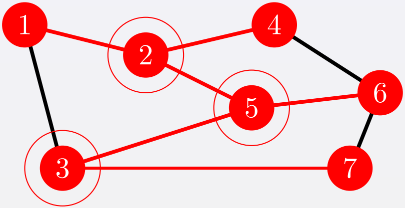
PTAS for m-SROCT

Optimal k-Star

Optimal k-Star

k-star

Given a graph G and a positive integer k, a k-**star** of G is a spanning tree T of G with no more than k internal nodes.
A **core** τ of a k-star T of G is a tree resulting by eliminating $n-k$ leaves from T.
Observe, $T = (\tau, S = \{S_{u_1}, ..., S_{u_k}\})$.



$$\tau = T[\{2,3,5\}]$$

$$S = \left\{ \begin{array}{l} S_2 = \{1,4\}, \\ S_3 = \{7\}, \\ S_5 = \{6\} \end{array} \right\}$$

Optimal k-Star

Configuration of k-star

Given a k-star $T = (\tau, S)$ a **configuration** of T is $(\tau, L = \{l_{u_1}, ..., l_{u_k}\})$ where $l_{u_i} = |S_{u_i}|$.

$$\left(\tau = T\left[\{2,3,5\}\right], S = \left\{\ S_2 = \{1,4\}, S_3 = \{7\}, S_5 = \{6\}\ \right\}\right)$$

$$\left(\tau = T\left[\{2,3,5\}\right], L = \left\{\ l_2 = 2, l_3 = 1, l_5 = 1\ \right\}\right)$$

Optimal k-Star

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Observe

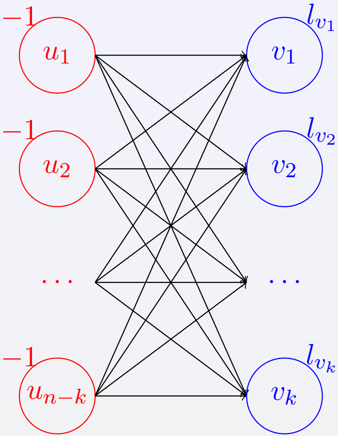
The number of possibles k-stars is exponential in n, while the number of possible configuration is $O\left(k^k n^{2k-1}\right)$ (Wu et al. (1998)).

Optimal k-Star

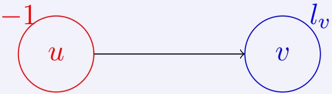
Obtaining an Optimal k-star

- For each possible configuration, select an optimal k-star of that configuration, returning the optimal among them.
- The problem of finding an optimal k-star of a given configuration can be reduced to Uncapacitated Minimum Cost Flow problem (UMCF).

Optimal k-Star



Optimal k-Star



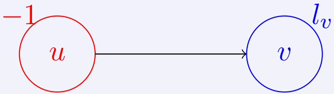
$$(R - r(u))\omega(u, v)$$

$$+r(u)l_v\omega(u, v)$$

$$+\sum_{w \in V_\tau - v} r(u)\left(l_w + 1\right)\left(\omega(u, v) + d(\tau, v, w)\right)$$

$$R = \sum_{u \in V} r(u)$$

Optimal k-Star



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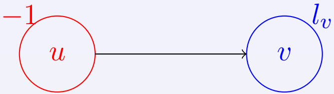
Optimal k-Star



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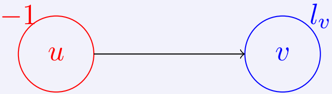
Optimal k-Star



$$(R - r(u))\omega(u,v)$$

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$$+\sum_{w\in V_\tau - v} r(u)\,(l_w+1)\,(\omega(u,v)+d(\tau,v,w))$$

$$R = \sum_{u\in V} r(u)$$

Optimal k-Star

Reduction to UMCF

- $V_{G'} = V_G$
- $E_{G'} = \{(u,v) | u \in V_{G-\tau} \wedge v \in \tau\}$
- $\omega'(u,v) = R\omega(u,v) + \sum_{w \in V_\tau} r(u)\left(d(\tau,v,w) + \omega(u,v)\right)(l_w + 1) - 2r(u)\omega(u,v)$
- $r'(u) = -1$ if $u \in V_{G-\tau}$, otherwise $r'(u) = l_u$.

Relation between Solutions Costs

$C(S') = C(S) - \sum_{u \in V_\tau} \sum_{v \in V_\tau} r(v)d(\tau,u,v)(l_u + 1)$

Optimal k-Star

Lemma 1

The optimum k-star for m-SROCT with fixed k can be solve in $O(n^{2k+1}\log^2(n))$ time.

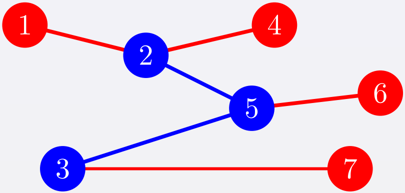
PTAS for m-SROCT

Reduction to Optimal k-star

Reduction to Optimal k-star

δ-separators

Given $0 < \delta \leq \frac{1}{2}$ and a spanning tree T of G, a sub-tree S of T is a δ-η-**separator** of T if every component B of $T - S$, satisfies $n(B) \leq \delta n$. If every component B of $T - S$, satisfies $r(B) \leq \delta R$, S is a δ-ρ-separator of T. If both conditions apply, S is a δ-$\eta\rho$-**separator** of T.



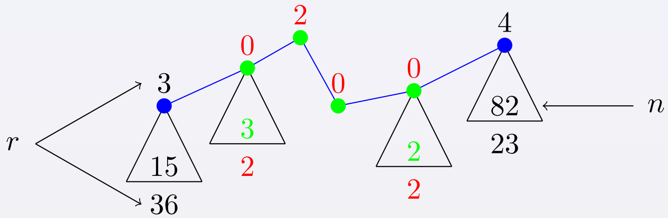
$\frac{1}{7}$-η-separator.

Time:

Reduction to Optimal k-star

δ-$\eta\rho$-paths

Given $0 < \delta \leq \frac{1}{2}$ and a spanning tree T of G, a path P of T is a δ-$\eta\rho$-**path** of T if $\eta_P^m \leq \delta \frac{n}{6}$ and $\rho_P^m \leq \delta \frac{R}{6}$.

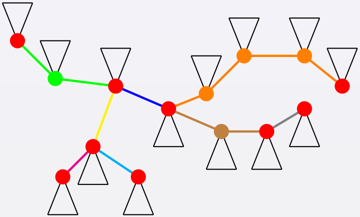


$\eta_P^m = 9 \leq \frac{1}{2}\frac{n=108}{6}$, $\rho_P^m = 6 \leq \frac{1}{2}\frac{R=72}{6} \Rightarrow P$ is a $\frac{1}{2}$-$\eta\rho$-path

Reduction to Optimal k-star

δ-spines

As set Y of internally disjoint δ-$\eta\rho$-paths is a δ-$\eta\rho$-spine of a spanning tree T of G if the union of its elements results in a minimal δ-$\eta\rho$-separator of T.
$ext(Y)$ denotes the endpoints set of all paths in Y.



Reduction to Optimal k-star

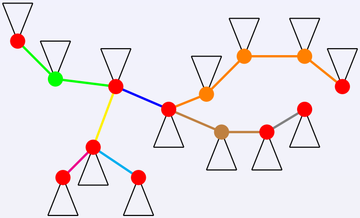
Lemma 2

Given $0 < \delta \leq \frac{1}{2}$, a spanning tree T of G and a δ-$\eta\rho$-spine Y of T, there exists a $|ext(Y)|$-star X of G satisfying $C(X) \leq \frac{1}{1-\delta}C(T)$.
The core of X lies on $ext(Y)$.

Reduction to Optimal k-star

δ-$\eta\rho$-spine Y of T

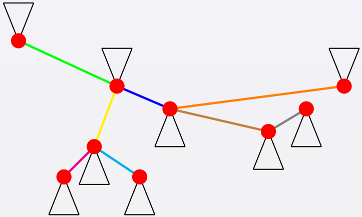
$|ext(Y)| = 9$



$|ext(Y)|$-star X

constructed from Y

$C(X) \leq \frac{1}{1-\delta} C(T)$



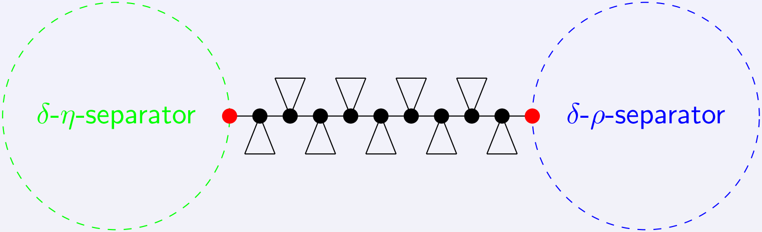
Reduction to Optimal k-star

Lemma 3

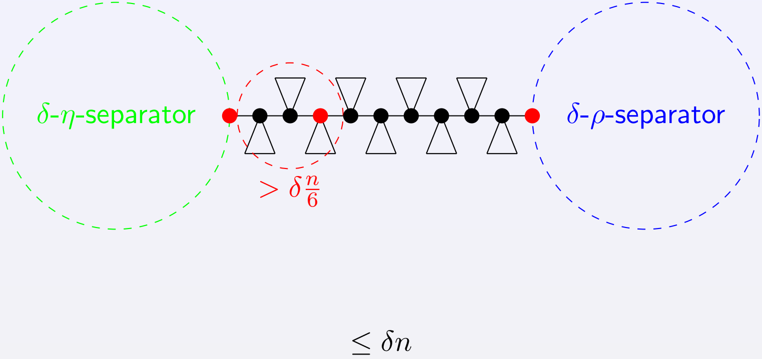
Given $0 < \delta \leq \frac{1}{2}$ and a spanning tree T of G, there exists a δ-$\eta\rho$-spine Y of T satisfying $|ext(Y)| \leq 3\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right)$.

We obtain Y from two sets Y_η and Y_ρ of δ-$\eta\rho$-paths, where the union of the elements in Y_η results in a δ-η-separator of T and the union of the elements in Y_ρ results in a δ-ρ-separator of T.

Reduction to Optimal k-star



Reduction to Optimal k-star



Reduction to Optimal k-star

Theorem 1

There exists a PTAS for m-SROCT, such that a $\left(1+\frac{\delta}{1-\delta}\right)$-approximation can be found in $O\left(n^{6\left(\left\lceil\frac{6}{\delta}\right\rceil^2-11\left\lceil\frac{6}{\delta}\right\rceil+1\right)+1}\log^2(n)\right)$ time complexity where $0<\delta\leq\frac{1}{2}$.

Reduction to Optimal k-star

proof

Consider $0 < \delta \leq \frac{1}{2}$ and T^* an optimal solution for m-SROCT.

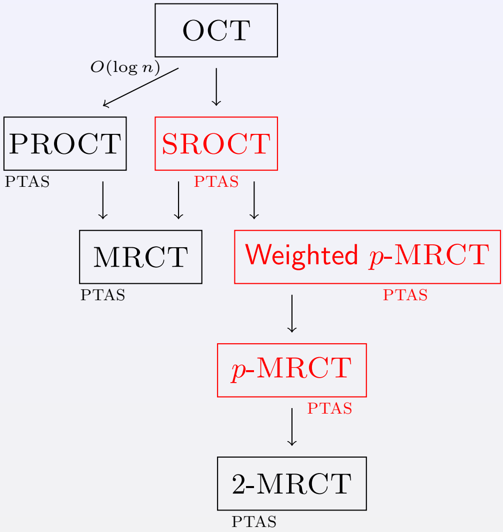
- From Lemma 3, there exists a there exists a δ-$\eta\rho$-spine Y of T^* such that $|ext(Y)| \leq 3\left(\left\lceil\frac{6}{\delta}\right\rceil^2 - 11\left\lceil\frac{6}{\delta}\right\rceil + 1\right)$;
- from Lemma 2, there exists a $|ext(Y)|$-star X such that $C(X) \leq \frac{1}{1-\delta}C(T^*)$;
- an optimal $|ext(Y)|$-star X^* satisfies $C(X^*) \leq C(X) \leq \frac{1}{1-\delta}C(T^*)$, and by Lemma 1 X^* can be found in $O(n^{2|ext(Y)|+1}\log^2(n))$ time.

Conclusions

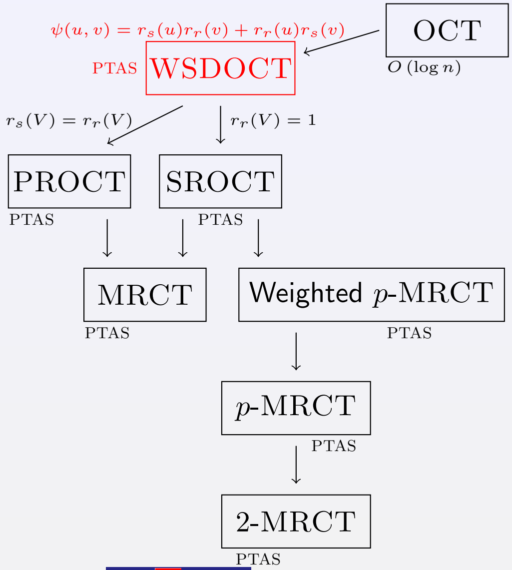
Conclusions

Problem	Best Result in Lit.		Our Approach	
	Ratio	Time Complexity	Ratio	Time Complexity
SROCT	2	$O\left(n^3\right)$	PTAS	$O\left(n^{6\left(\left\lceil\frac{6}{\delta}\right\rceil^2-11\left\lceil\frac{6}{\delta}\right\rceil+1\right)+2}\right)$
MRCT	PTAS	$O\left(n^{2\left\lceil\frac{2}{\delta}\right\rceil-2}\right)$	PTAS	$O\left(n^{6\left(\left\lceil\frac{6}{\delta}\right\rceil^2-11\left\lceil\frac{6}{\delta}\right\rceil+1\right)+2}\right)$
weighted p-MRCT	2	$O\left(n^3\right)$	PTAS	$O\left(n^{6\left(\left\lceil\frac{6}{\delta}\right\rceil^2-11\left\lceil\frac{6}{\delta}\right\rceil+1\right)+2}\right)$
p-MRCT	2	$O\left(n^3\right)$	PTAS	$O\left(n^{6\left(\left\lceil\frac{6}{\delta}\right\rceil^2-11\left\lceil\frac{6}{\delta}\right\rceil+1\right)+2}\right)$
2-MRCT	PTAS	$O\left(n^{\frac{1}{\delta}+1}\right)$	PTAS	$O\left(n^{6\left(\left\lceil\frac{6}{\delta}\right\rceil^2-11\left\lceil\frac{6}{\delta}\right\rceil+1\right)+2}\right)$

Conclusions



Conclusions



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Thanks for your attention!
Questions?