AXIOMATIC CHARACTERIZATION OF THE MEDIAN AND ANTIMEDIAN FUNCTIONS ON COCKTAIL-PARTY GRAPHS AND COMPLETE GRAPHS

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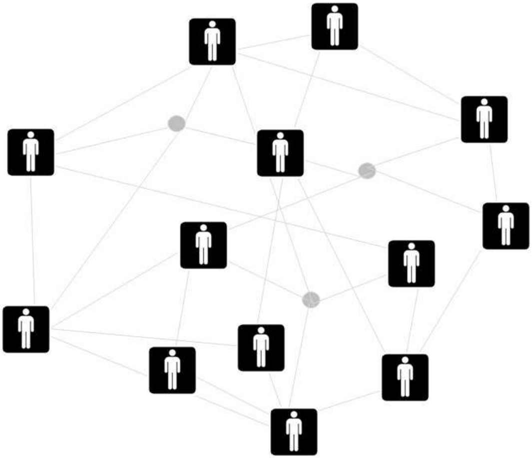
CALDAM 2015 @ IIT Kanpur, 9 Feb 2015

OVERVIEW

PLAN

SCENARIO



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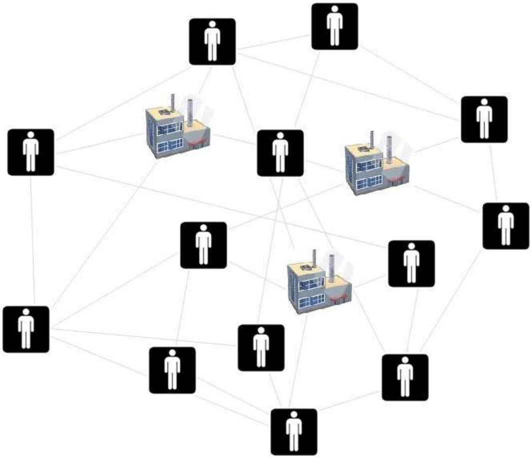
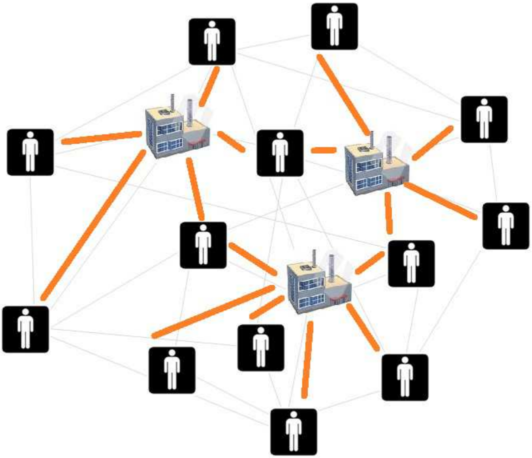


Scenario



Scenario - Different types of facilities

Common facility

Emergency facility

Obnoxious facility

Motivation

Facility Location Problem

MOTIVATION

Facility Location Problem

- Optimization problem

Motivation

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- Optimization problem
- Locating a service facility optimally

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Client Utility

- A monotone function of distance the client has to travel to reach the facility.

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Optimality Criterion

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1. **Mean** Location that minimizes the sum of the squares of the distances to the clients.

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- **Rationality guaranteed** The function should satisfy certain "rational" rules.

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K. Arrow initiated the study of the axiomatics of consensus functions in 1951 [1].

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A function with nice properties might be characterized by simple axioms.

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1. Optimality criterion which is the function under study.
2. Structure of network in which function is defined.
3. Continuous or discrete structure.

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Axiomatic Characterization of Location Function

1. Optimality criterion depends on type of facility.
2. Structure of network represented as a graph and clients/facilities are required to be located at vertices only.
3. Structure is discrete.

Location Function	Structure	Author	Year
Mean function	Tree networks	Holzman [10]	1990
Mean function	Trees (discrete case)	McMorris et.al. [14, 15]	2010, 2012
Median function	Tree networks (continuous case)	Vohra [26]	1996
Median function	Cube-free median graphs	McMorris et.al.	1998
Median function	Hypercubes and median graphs	Mulder and Novick [22], [23]	2011, 2013

Location Function	Structure	Author	Year
Center function	Trees	McMorris et.al. [18] and Mulder et.al. [24]	2001, 2008
Antimedian function	Hypercubes and paths	Balakrishnan et.al. [8]	2012

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  1. Antimedian function on complete graphs
  2. Median and antimedian functions on cocktail-party graphs

PLAN

Distance Function and Interval

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Interval

Interval $I(u, v)$ between two vertices u and v in G consists of all vertices on shortest u, v-paths, that is:

$$I(u, v) = \{x \mid d(u, x) + d(x, v) = d(u, v)\} \quad (1)$$

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- π^m The profile consisting of the concatenation of m copies of π.

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- For convenience, we write $F(x_1, \ldots, x_k)$ instead of $F((x_1, \ldots, x_k))$.

Remoteness function

The *remoteness* of a vertex v to profile π is defined as

$$r(v, \pi) = \sum_{i=1}^{k} d(x_i, v). \tag{2}$$

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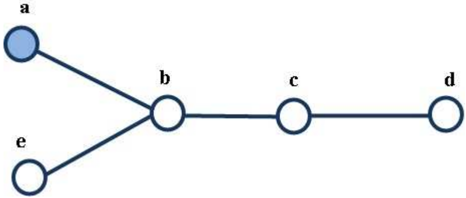
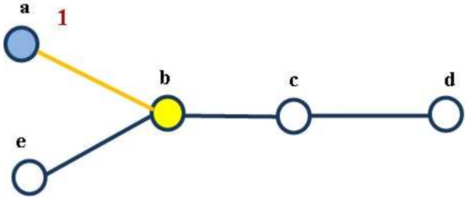
Remoteness Computation - Finding remoteness of vertex a



Remoteness Computation - Finding remoteness of vertex *a*



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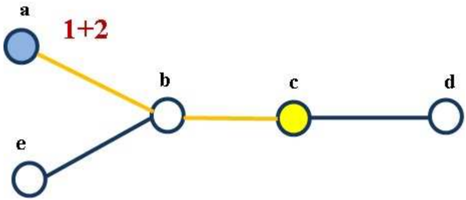

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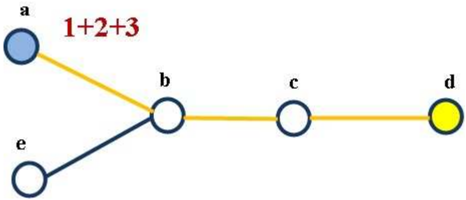

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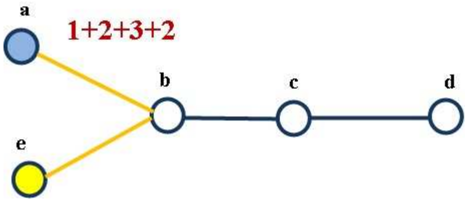
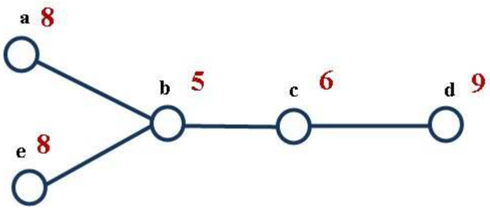


Remoteness Computation - Finding Median and Antimedian



Median $M(G) = \{b\}$ Antimedian $AM(G) = \{d\}$

MEDIAN OF PROFILES OF LENGTH 1, 2 AND 3

$$M(x) = \{x\}, \tag{3}$$

$$M(x, y) = I(x, y). \tag{4}$$

If $I(u, v) \cap I(v, w) \cap I(w, u) \neq \emptyset$, then

$$M(u, v, w) = I(u, v) \cap I(v, w) \cap I(w, u). \tag{5}$$

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Median Graph

For any three vertices u, v, w, $|I(u, v) \cap I(v, w) \cap I(w, u)| = 1$.
Any profile of length 3 has a unique median.

Commonly Used Axioms on Median Function

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(A) Anonymity: $F(\pi) = F(x_{\chi(1)}, x_{\chi(2)}, \ldots, x_{\chi(k)})$, for any profile $\pi = (x_1, x_2, \ldots, x_k)$ on V and for any permutation χ of $\{1, 2, \ldots, k\}$.

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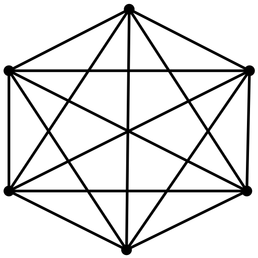
The first two axioms are defined without any reference to metric.

Plan

Complete Graphs

Complete Graph K_6



ANTIMEDIAN FUNCTION

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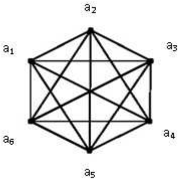
- $AM(\pi) = W_\pi$

COMPUTING $AM(\pi)$

Graph K_6



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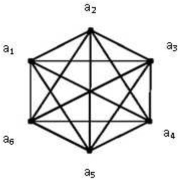
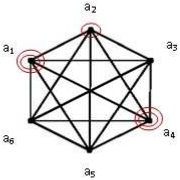
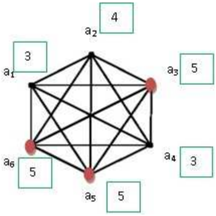


Profile $\pi = (a_1, a_2, a_4, a_1, a_4)$



Computing $AM(\pi)$

$$AM(\pi) = W_\pi = \{a_3, a_5, a_6\}$$



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Completeness: $F(v_1, v_2, \ldots, v_n) = V$.

Complement: $F(x) = V - \{x\}$, for each $x \in V$.

4 Axioms

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Theorem 4.1

Let F be a consensus function on K_n with $n > 1$. Then F is the antimedian function if and only if F satisfies (A), (C), Completeness and Complement.

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Proof.

$AM(\pi)$ satisfies the four axioms. Conversely, let F satisfy the four axioms.
Let $\pi = (x_1, x_2, \ldots, x_k)$.
$\{\pi\} \subset V$: π = concatenation of $(x_1), (x_2), \ldots, (x_k)$. By *Complement*: $F(x_1) \cap \ldots \cap F(x_k) = W_\pi = V - \{\pi\}$. (So (C))
All $v \in V$ occur exactly m times in π ($m > 0$). Then $\pi = (v_1, v_2, \ldots, v_n)^m$ due to (A). (So (C) and *Completeness*).
π is any other profile. Some vertices occur exactly m times in π and other vertices occur more than m times.
$\pi = \pi'(v_1, v_2, \ldots, v_n)^m$ due to (A) [$W_{\pi'}$ is the set of vertices that occur exactly m times in π]
By the above observations and (C), we have $F(\pi) = F(\pi') \cap V = W_{\pi'} = AM(\pi)$. □ □

Independence of axioms

We do not yet have an example that shows whether *Anonymity* is independent from the other axioms.

INDEPENDENCE OF AXIOM *Complement*

Complement excluded.

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- Let F be defined by $F(\pi) = V$ for all profiles.

Independence of axiom *Complement*

Complement excluded.

- Let F be defined by $F(\pi) = V$ for all profiles.
- Then it fails Complement but satisfies trivially the other axioms.

Independence of axiom (C)

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  $(k1)$: $F(x) = V - \{x\}$, for any $x \in V$,
  $(k2)$: $F(\pi) = V$, for any profile π of length at least 2.

INDEPENDENCE OF AXIOM (C)

(C) excluded.

- Let F be defined by
  $(k1)$: $F(x) = V - \{x\}$, for any $x \in V$,
  $(k2)$: $F(\pi) = V$, for any profile π of length at least 2.
- Then F fails (C) but trivially satisfies the other axioms.

Independence of axiom *Completeness*

Completeness excluded.

INDEPENDENCE OF AXIOM *Completeness*

Completeness excluded.

- Let F be defined by

INDEPENDENCE OF AXIOM *Completeness*

Completeness excluded.

- Let F be defined by

  (*k*3): $F(\pi) = \{v_1\}$, for any π with $\{\pi\} = V$,

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Completeness excluded.

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  $(k3)$: $F(\pi) = \{v_1\}$, for any π with $\{\pi\} = V$,

  $(k4)$: $F(\pi) = V - \{\pi\}$, for any π with $\{\pi\} \neq V$.

INDEPENDENCE OF AXIOM *Completeness*

Completeness excluded.

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INDEPENDENCE OF AXIOM *Completeness*

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- F satisfies (A) and *Complement*.

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- By $(k3)$ F fails *Completeness*.

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Completeness excluded.

- Let F be defined by
  $(k3)$: $F(\pi) = \{v_1\}$, for any π with $\{\pi\} = V$,

  $(k4)$: $F(\pi) = V - \{\pi\}$, for any π with $\{\pi\} \neq V$.

- F satisfies (A) and *Complement*.
- By $(k3)$ F fails *Completeness*.
- Check *Consistency*

Let π and ρ be two profiles.

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$\{\pi\} = V = \{\rho\}$

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$\{\pi\} = V$ and $\{\rho\} \neq V$

If ρ does not contain v_1, then $F(\pi) \cap F(\rho) \neq \emptyset$.

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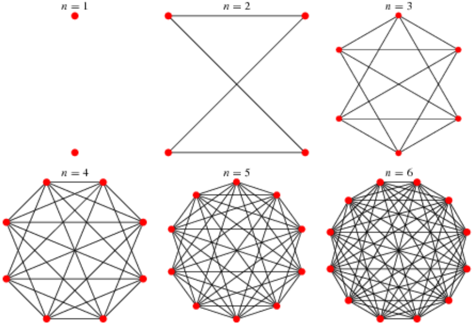
- Again we have $F(\pi\rho) = F(\pi) \cap F(\rho)$.

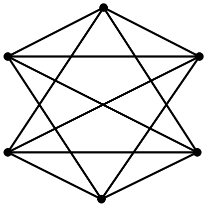
Plan

Cocktail-party Graphs



Definitions

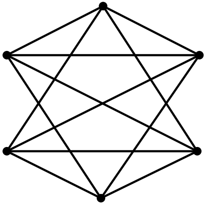
Cocktail-party graph $K_{(n\times 2)}$



DEFINITIONS

Cocktail-party graph $K_{(n\times 2)}$

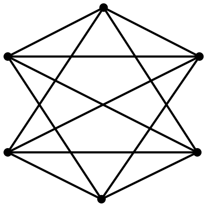
- Complete graph K_{2n} with $V = \{v_1, \dots, v_n, v_{n+1}, \dots, v_{2n}\}$.



DEFINITIONS

Cocktail-party graph $K_{(n \times 2)}$

- Complete graph K_{2n} with $V = \{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$.
- Delete perfect matchings $v_1 v_{n+1}$, ..., $v_n v_{2n}$.



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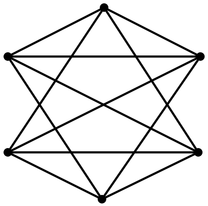

Definitions

Cocktail-party graph $K_{(n \times 2)}$

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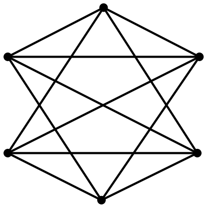
- Delete perfect matchings $v_1 v_{n+1}$, ..., $v_n v_{2n}$.
- Arises in the handshake problem.
- Distance-transitive, and hence also Distance-regular.



DEFINITIONS

Mating pairs

Definitions

Mating pairs

- **Mates** $(v, \tilde{v})$: $\{v_i, v_{n+i}\}$ $(1 \leq i \leq n)$ is a pair of *mates*.

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- $\tilde{\pi}$: Obtained from π by replacing each element by its mate.
- **Mating pair**: The profile $(v, \tilde{v})$.
- **Mating profile**: The concatenation of mating pairs.

Remoteness

Lemma 5.1

Let G be a cocktail-party graph with vertex set V, and let $\pi = (v, \tilde{v})$ be a mating pair. Then $r(u, \pi) = 2$, for all v in V.

Computing median and antimedian

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- π : a profile on the cocktail-party graph.

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- π : a profile on the cocktail-party graph.
- π' : $\pi \setminus (v, \tilde{v})$.

COMPUTING MEDIAN AND ANTIMEDIAN

- π : a profile on the cocktail-party graph.
- π' : $\pi \setminus (v, \tilde{v})$.
- $M(\pi) = M(\pi')$ and $AM(\pi) = AM(\pi')$.

Computing median and antimedian

- π : a profile on the cocktail-party graph.
- π' : $\pi \setminus (v, \tilde{v})$.
- $M(\pi) = M(\pi')$ and $AM(\pi) = AM(\pi')$.
- Mate-free subprofile ρ (after deleting all mates in π)

Computing median and antimedian

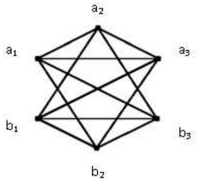
- π : a profile on the cocktail-party graph.
- π' : $\pi \setminus (v, \tilde{v})$.
- $M(\pi) = M(\pi')$ and $AM(\pi) = AM(\pi')$.
- Mate-free subprofile ρ (after deleting all mates in π)
- $M(\pi) = Pl(\rho)$ (the vertices with highest occurrence in ρ,)

COMPUTING MEDIAN AND ANTIMEDIAN

- π : a profile on the cocktail-party graph.
- π' : $\pi \setminus (v, \tilde{v})$.
- $M(\pi) = M(\pi')$ and $AM(\pi) = AM(\pi')$.
- Mate-free subprofile ρ (after deleting all mates in π)
- $M(\pi) = Pl(\rho)$ (the vertices with highest occurrence in ρ,)
- $AM(\pi) = Pl(\tilde{\rho})$ (the mates of the vertices with highest occurrence)

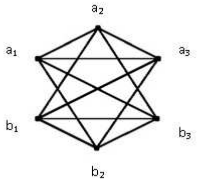
Computing $M(\pi)$ and $AM(\pi)$

Graph $K_{3\times 2}$

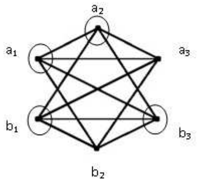


COMPUTING $M(\pi)$ AND $AM(\pi)$

Graph $K_{3\times 2}$

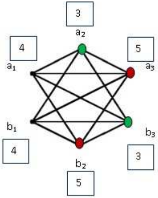


Profile $\pi = (a_1, b_1, a_2, b_3)$



Computing $M(\pi)$ and $AM(\pi)$

$M(\pi) = \{a_2, b_3\}$ and $AM(\pi) = \{b_2, a_3\}$



Median Function

Lemma 5.2

Let F be the median function defined on the vertex set V of a cocktail-party graph G. Then $F(v, \tilde{v}) = V$, for any $v \in V$.

Median Function

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Proof.

Direct consequence of Lemma 5.1. □

Median Function

Lemma 5.3

Let F be the median function defined on the vertex set V of a cocktail-party graph G. Then $F(\pi) = Pl(\pi)$*, for all mate-free profiles* π.

Median Function

Lemma 5.3

Let F be the median function defined on the vertex set V of a cocktail-party graph G. Then $F(\pi) = Pl(\pi)$, for all mate-free profiles π.

Proof.

$\pi = (x_1, x_2, \ldots, x_k)$ be a mate-free profile.
$\{\pi\} = \{y_1, y_2, \ldots, y_\ell\}$
f_j the number of occurrences of y_j in π.
For any vertex w outside the profile π, we have $d(w, y_j) \geq 1$, for each vertex y_j in π.
$f = \sum_{j=1}^{\ell} f_j$.
$r(w, \pi) \geq f$.
u be any vertex in π.
$d(u, x_i) = 1$, for any $x_i \neq u$.
$r(u, \pi) = f - f_j$, for $u = y_j$. (vertices that minimize remoteness are all in π).
$r(u, \pi) = f - f_j$ is minimum when f_j is maximum. (vertices that minimize remoteness are precisely those that occur most often in π). □ □

Axioms by Lemma 5.2 and Lemma 5.3

(A_1): $F(v, \tilde{v}) = V$, for all $v \in V$.

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(A_1): $F(v, \tilde{v}) = V$, for all $v \in V$.

(A_2): $F(\pi) = Pl(\pi)$, for all mate-free profiles π.

Remark 5.4

Let F be a consensus function defined on the vertex set V of a cocktail-party graph G such that F satisfies A_1 and A_2. Then F satisfies the Betweenness axiom (B).

4 AXIOMS

Theorem 5.5

Let F be a consensus function on a cocktail-party graph G with vertex set V. Then F is the median function if and only if F satisfies axioms (A), (C), (A_1) and (A_2).

4 Axioms

Theorem 5.5

Let F be a consensus function on a cocktail-party graph G with vertex set V. Then F is the median function if and only if F satisfies axioms (A), (C), (A_1) and (A_2).

Proof.

The median function satisfies all the four axioms.
F a function satisfying the four axioms.
If π contains a pair of mates $v, \tilde{v}$, we can permute π as $(v, \tilde{v})\rho$
ρ is the subprofile of π obtained by deleting the elements v and $\tilde{v}$ from their respective positions.
(A_1): $F(v, \tilde{v}) = V$.
So $F(v, \tilde{v}) \cap F(\rho) \neq \emptyset$.
By (C), $F((v, \tilde{v})\rho) = F(v, \tilde{v}) \cap F(\rho) = F(\rho)$.
By (A), $F(\pi) = F(v, \tilde{v}) \cap F(\rho) = F(\rho)$.
Repeat this process until we end up with a subprofile σ of π that is either a mating pair or mate-free.
If σ is mate-free, $F(\pi) = F(\sigma)$. By (A_2), $F(\sigma) = Pl(\sigma) = M(\sigma) = M(\pi)$.
If σ is a mating pair, $F(\sigma) = V = F(\pi) = M(\pi)$. □ □

Independence of axioms

Examples

Independence of axioms

Examples

- We want to know whether the axioms involved are independent.

Independence of axioms

Examples

- We want to know whether the axioms involved are independent.
- In all examples G is a cocktail-party graph with vertex set V having at least 4 vertices.

Independence of axiom (A_1)

(A_1) excluded.

INDEPENDENCE OF AXIOM (A_1)

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- Define $F(\pi) = Pl(\pi)$, for all profiles π.

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- Define $F(\pi) = Pl(\pi)$, for all profiles π.
- F satisfies (A), (C) and (A_2).

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(A_1) excluded.

- Define $F(\pi) = Pl(\pi)$, for all profiles π.
- F satisfies (A), (C) and (A_2).
- Since $F(v, \tilde{v}) = \{v, \tilde{v}\} \neq V$, for any vertex v, the function F does not satisfy (A_1).

Independence of axiom (A_2)

(A_2) excluded.

INDEPENDENCE OF AXIOM (A_2)

(A_2) excluded.

- Define $F(\pi) = V$, for all profiles π.

Independence of axiom (A_2)

(A_2) excluded.

- Define $F(\pi) = V$, for all profiles π.
- F satisfies (A), (C) and (A_1).

INDEPENDENCE OF AXIOM (A_2)

(A_2) excluded.

- Define $F(\pi) = V$, for all profiles π.
- F satisfies (A), (C) and (A_1).
- For any two adjacent vertices u and v in G,

$$F(u, v) = V \neq \{u, v\} = Pl(u, v). \tag{6}$$

So F does not satisfy (A_2).

Independence of axiom (C)

(C) excluded.

Independence of axiom (C)

(C) excluded.

- Define

Independence of axiom (C)

(C) excluded.

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Independence of axiom (C)

(C) excluded.

- Define

  $(c1)$: $F(v, \tilde{v}) = V$, for all vertices v in V,
  $(c2)$: $F(\pi) = Pl(\pi)$, for all profiles π that are not a mating pair.

Independence of axiom (C)

(C) excluded.

- Define

  $(c1)$: $F(v, \tilde{v}) = V$, for all vertices v in V,
  $(c2)$: $F(\pi) = Pl(\pi)$, for all profiles π that are not a mating pair.

- F satisfies (A), (A_1) and (A_2).

Independence of axiom (C)

(C) excluded.

- Define
  $(c1)$: $F(v, \tilde{v}) = V$, for all vertices v in V,
  $(c2)$: $F(\pi) = Pl(\pi)$, for all profiles π that are not a mating pair.
- F satisfies (A), (A_1) and (A_2).
- Take two vertices u and v that are not mates, and let $\pi = (u, \tilde{u}, v, \tilde{v})$.

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(C) excluded.

- Define
  $(c1)$: $F(v, \tilde{v}) = V$, for all vertices v in V,
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- F satisfies (A), (A_1) and (A_2).
- Take two vertices u and v that are not mates, and let $\pi = (u, \tilde{u}, v, \tilde{v})$.
- by $(c2)$, we have

$$F(\pi) = Pl(\pi) = \{u, \tilde{u}, v, \tilde{v}\} \neq V = F(u, \tilde{u}) \cap F(v, \tilde{v}). \quad (7)$$

So F does not satisfy *Consistency*.

Independence of axiom (A)

The case of *Anonymity*

INDEPENDENCE OF AXIOM (A)

The case of *Anonymity*

- The independence of *Anonymity* is a non-trivial issue.

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- Not to expect that it follows from the other axioms.

INDEPENDENCE OF AXIOM (A)

The case of *Anonymity*

- The independence of *Anonymity* is a non-trivial issue.
- We do not yet have an example that shows independence of *Anonymity*.
- Not to expect that it follows from the other axioms.
- Open problem here.

Antimedian Function-Axioms

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(A_3): $F(\pi) = Pl(\tilde{\pi})$, for all mate-free profiles π.

Antimedian Function-Axioms

(A_3): $F(\pi) = Pl(\tilde{\pi})$, for all mate-free profiles π.

Theorem 5.6

Let F be a consensus function on a cocktail-party graph G with vertex set V. Then F is the antimedian function if and only if F satisfies axioms (A), (C), (A_1) and (A_3).

Independence of axioms

Examples

Independence of axioms

Examples

- We want to know whether the axioms involved are independent.

Independence of axioms

Examples

- We want to know whether the axioms involved are independent.
- In all examples G is a cocktail-party graph with vertex set V having at least 4 vertices.

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INDEPENDENCE OF AXIOM (A_1)

(A_1) excluded.

- $F(\pi) = Pl(\tilde{\pi})$, for all profiles π.
- F satisfies (A), (C) and (A_3).
- Since $F(v, \tilde{v}) = \{\tilde{v}, v\} \neq V$, for any vertex v, the function F does not satisfy (A_1).

Independence of axiom (A_3)

(A_3) excluded.

INDEPENDENCE OF AXIOM (A_3)

(A_3) excluded.

- $F(\pi) = V$, for all profiles π.

INDEPENDENCE OF AXIOM (A_3)

(A_3) excluded.

- $F(\pi) = V$, for all profiles π.
- F satisfies axioms (A), (C) and (A_1).

INDEPENDENCE OF AXIOM (A_3)

(A_3) excluded.

- $F(\pi) = V$, for all profiles π.
- F satisfies axioms (A), (C) and (A_1).
- Take any two adjacent vertices u and v in G. Then

$$F(u, v) = V \neq \{\tilde{u}, \tilde{v}\} = Pl(\tilde{u}, \tilde{v}). \tag{8}$$

Independence of axiom (A_3)

(A_3) excluded.

- $F(\pi) = V$, for all profiles π.
- F satisfies axioms (A), (C) and (A_1).
- Take any two adjacent vertices u and v in G. Then

$$F(u, v) = V \neq \{\tilde{u}, \tilde{v}\} = Pl(\tilde{u}, \tilde{v}). \quad (8)$$

- So F does not satisfy (A_3).

Independence of axiom (C)

(C) excluded.

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- Define the function F on G by

Independence of axiom (C)

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- Define the function F on G by

  $(c1)$ $F(v, \tilde{v}) = V$, for all vertices v in V,

Independence of axiom (C)

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- Define the function F on G by
  - $(c1)$ $F(v, \tilde{v}) = V$, for all vertices v in V,
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- Define the function F on G by

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- Take two vertices u and v that are not mates, and let $\pi = (u, \tilde{u}, v, \tilde{v})$.

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- Define the function F on G by
  (c1) $F(v, \tilde{v}) = V$, for all vertices v in V,
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- F satisfies (A), (A_1) and (A_3).
- Take two vertices u and v that are not mates, and let $\pi = (u, \tilde{u}, v, \tilde{v})$.
- Then, by (c2), we have

$$F(\pi) = Pl(\tilde{\pi}) = \{\tilde{u}, u, \tilde{v}, v\} \neq V = F(u, \tilde{u}) \cap F(v, \tilde{v}). \tag{9}$$

INDEPENDENCE OF AXIOM (C)

(C) excluded.

- Define the function F on G by
  $(c1)$ $F(v, \tilde{v}) = V$, for all vertices v in V,
  $(c2)$ $F(\pi) = Pl(\tilde{\pi})$, for all profiles π that are not a mating pair.
- F satisfies (A), (A_1) and (A_3).
- Take two vertices u and v that are not mates, and let $\pi = (u, \tilde{u}, v, \tilde{v})$.
- Then, by $(c2)$, we have

$$F(\pi) = Pl(\tilde{\pi}) = \{\tilde{u}, u, \tilde{v}, v\} \neq V = F(u, \tilde{u}) \cap F(v, \tilde{v}). \tag{9}$$

- So F does not satisfy *Consistency*.

Independence of axiom (A)

We do not have an example yet that shows the independency of *Anonymity*. We leave this as an open problem.

Plan

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THANK YOU