On Spectra of Corona Graphs

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Outline

Introduction

Product graphs have been used to generate mathematical models[1] of complex networks.

- What are Complex networks?
  - Defined as networks with non-trivial topology and dynamics.
  - Their ubiquity suggests that they are the skeletons of complex systems in the real-world.
  - Examples
    - Biological Networks like food web
    - Transportation Network like railway network
    - Social Networks
    - Telecommunications
    - Informational networks

We propose a method of network generation for modelling complex networks which we call corona graphs.

We investigated the spectra, Laplacian spectra and signless Laplacian spectra of corona graphs to find their topological properties.

[1] Leskovec, Jure and Chakrabarti, Deepayan and Kleinberg, Jon and Faloutsos, Christos and Ghahramani, Zoubin.: Kronecker graphs: An approach to modeling networks. J. Mach. Learn. Res. 11, 985–1042 (2010).

Outline

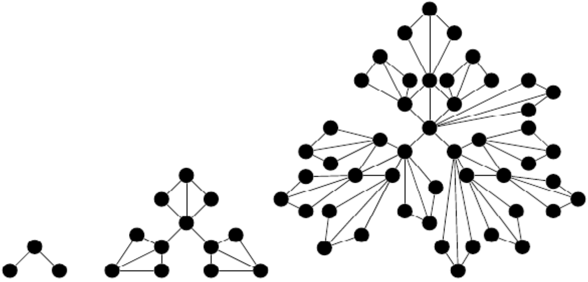
Corona Graphs

Let $G = (V, E)$ be a graph having set of nodes $V = \{v_1, \dots, v_n\}$ such that $|V| = n$, and E the set of edges. The adjacency matrix $A(G) = [a_{v_i v_j}]$ of G having dimension $|V| \times |V|$ is defined by $a_{v_i, v_j} = 1$ if $(v_i, v_j) \in E$ otherwise $a_{v_i, v_j} = 0$ when columns and rows are labelled by nodes of G.

- The corona product of the two graphs[1] G_1, G_2, denoted by $G_1 \circ G_2$ is obtained by taking an instance of G_1 and $|V_{G_1}|$ instances of G_2 and hence connecting the i^{th} node of G_1 to every node in the i^{th} instance of G_2 for each i.
- We extend this definition to define corona graphs. Let $G^{(0)} = G$. Given a basic graph G, the corona graphs generated by G are given by

$$G^{(m+1)} = G^{(m)} \circ G \qquad (1)$$

where $m(\geq 0)$ is a large natural number.

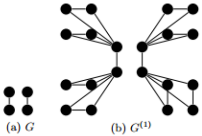


(a) P_3 (b) $G^{(1)}$ (c) $G^{(2)}$

[1] Frucht, Roberto and Harary, Frank.: On the corona of two graphs. Aequationes Math. 4, 322-325 (1970).

Corona Graphs

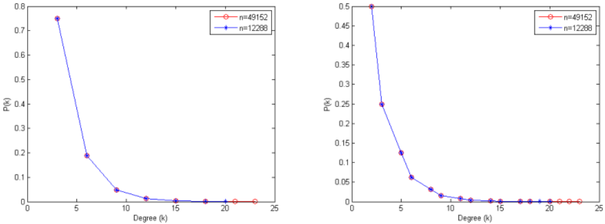
- Observations
  - The number of nodes in $G^{(m)}$ is
  $$|V^{(m)}| = n(n+1)^m. \quad (2)$$
  - If $|E|$ and n are the number of edges and nodes in the basic graph G respectively, then number of edges in $G^{(m)}$ is
  $$|E^{(m)}| = (|E| + (|E| + n)((n+1)^m - 1)). \quad (3)$$
  - Connectivity of $G^{(m)}$: Since G is connected, corona graphs generated by G are connected graphs. Evidently, if basic graph G is disconnected, it generates disconnected corona graphs.



Corona graphs

- Observations
  - Degree sequence of corona graphs: Assume that degree sequence of $G^{(0)} = G$ is given by $\{d_{i_1}^{(0)}, d_{i_2}^{(0)}, \ldots, d_{i_n}^{(0)}\}$ where $d_{i_l}^{(j)}$ represents the degree of node i_l at j^{th} corona product, and x is the total distinct degrees. The degree sequence for $G^{(m)}$ is obtained as $\{D_{i_j}^{(1)}, (D_{i_j}^{(2)}, \ldots, n \text{ times}), \ldots, (D_{i_j}^{(x)}, \ldots, n(n+1)^{m-1} \text{ times})\}$, where $D_{i_j}^{(1)} = d_{i_j} + mn$, $D_{i_j}^{(2)} = d_{i_j} + (m-1)n$, $\ldots, D_{i_j}^{(x)} = d_{i_j} + 1$.
  - Degree Distribution

$$P(k) = \frac{\sum_{j=1}^{n}\left(\delta_{k,D_{i_j}^{(1)}} + n\delta_{k,D_{i_j}^{(2)}} + n(n+1)\delta_{k,D_{i_j}^{(3)}} + \ldots + n(n+1)^{(m-1)}\delta_{k,D_{i_j}^{(x)}}\right)}{n(n+1)^m} \quad (4)$$



(a) For K_3 with $G^{(6)}$ and $G^{(7)}$ (b) For P_3 with $G^{(6)}$ and $G^{(7)}$

Outline

Graph spectra

Let G be a simple connected graph. The adjacency matrix $A(G^{(m)})$ associated with $G^{(m)}$ is given by

$$\mathbf{A}(\mathbf{G}^{(\mathbf{m})}) = \begin{bmatrix} A(G^{(m-1)}) & \mathbf{1}^T_{n(n+1)^{m-1}} \otimes I_{n(n+1)^{m-1}} \\ \mathbf{1}_{n(n+1)^{m-1}} \otimes I_{n(n+1)^{m-1}} & A(G) \otimes I_{n(n+1)^{m-1}} \end{bmatrix}$$

where $A(G^{(m-1)})$ is the adjacency matrix of $G^{(m-1)}$, $I_{n(n+1)^{m-1}}$ is the identity matrix and $\mathbf{1}_{n(n+1)^{m-1}}$ is the column vector of 1s of length $n(n+1)^{m-1}$. We denote spectra of corona graphs i.e. spectrum of $A(G^{(m)})$ by

$$\sigma(G^{(m)}) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n(n+1)^m}\} \tag{5}$$

where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n(n+1)^m}$ and spectral radius of $A(G^{(m)})$ is denoted as $\rho(G^{(m)})$.
The spectrum of corona product of two graphs $G = G_1 \circ G_2$ where G_1 is any graph and G_2 is a regular graph, and the Laplacian spectrum of corona product of any two graphs are provided by Barik et al.[1]
Inspired by their work, we derive $\sigma(G^{(m)})$ when the basic graph G is regular.

- Application of spectral radius
  - Smaller is its value,larger is robustness against propagation of viruses[2].
  - Helps in finding the community structure of the network.

[1] Barik, S and Pati, S and Sarma, BK.: The spectrum of the corona of two graphs. SIAM J. Discrete Math. 21, 47–56, 2007.

[2] A. Jamakovic, Characterization of complex networks: application to robustness analysis, PhD thesis, Delft University of Technology, 2008.

Graph spectra

Theorem

Let $G^{(0)} = G$ be a regular graph such that $\sigma(G) = (\mu_1, \mu_2, \ldots, \mu_n = r)$ (where, $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n = r$) and spectral radius of G be $\rho(G)$. Then, $\sigma(G^{(m)})$ is given by

(a) $\lambda_i = \dfrac{\mu_i + r\sum_{a=0}^{m-1} 2^a \pm \left(\sum_{c=1}^{m-1} z_c + \sqrt{((r-\mu_i) \pm \sum_{c=1}^{m-1} z_c)^2 + 2^{2m}.n}\right)}{2^m} \in \sigma(G^{(m)})$, *with multiplicity* 1 *for* $i = 1, \ldots, n(n+1)^{m-1}$
where,
$z_1 = \sqrt{(r-\mu_i)^2 + 4n}, \ldots, z_{m-1} = z_{m-2} + \sqrt{((r-\mu_i) \pm z_{m-2})^2 + n.2^{2(m-2)}}$.

(b) $\mu_i \in \sigma(G^{(m)})$, *with multiplicity* $n(n+1)^{m-1}$ *for* $i = 1, \ldots, n-1$.

The spectral radius of $G^{(m)}$ is given by

$$\rho(G^{(m)}) = \frac{\mu_n + r\sum_{a=0}^{m-1} 2^a + (\sum_{c=1}^{m-1} z_c + \sqrt{((r-\mu_i) - \sum_{c=1}^{m-1} z_c)^2 + 2^{2m}.n})}{2^m}$$

where z_c is defined above.

Graph spectra of star graphs

The following theorem gives the spectra of star graphs S_k with $k \geq 3$ as a special case of irregular graphs.

Theorem

Let $k \geq 3$ be an integer. The spectrum of the graph $S_k \circ S_k$ consists of the following eigenvalues

(a) $\lambda_z = X_1^z + \frac{\mu_i}{3} \in \sigma(G^{(1)})$ *with multiplicity* 1, *and*

(b) $0 \in \sigma(G^{(1)})$ *with multiplicity* $k(k-2)$.

where, λ_z *are the eigenvalues for* $G^{(1)}$ *with* $z = 1, 2, 3$ *for the* 3 *angles i.e.* $\frac{\theta}{3}, \frac{2\pi+\theta}{3}, \frac{4\pi+\theta}{3}$ *as shown in following sub-expressions*

$X^z = \frac{2}{3} \cos \frac{y\pi+\theta}{3} \sqrt{\mu_i^2 + (6k-3)}$, $\theta = \cos^{-1}\left(\frac{2\mu_i^3+\mu_i(18-9k)+(54k-54)}{2(\mu_i^2+(6k-3))^{\frac{3}{2}}}\right) - y\pi$

where $y = 0, 2, 4$. *Here,*

$\lambda_z \in \left[-\mu_i + \frac{2\mu_i}{3} - \frac{2\sqrt{(6k-3)}}{3}, \mu_i + \frac{2\sqrt{(6k-3)}}{3}\right]$

Graph spectra of star graphs

Corollary

Let G be the basic star graph S_k for each $k \geq 3$ such that $\sigma(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$. Let $m \geq 1$.Then $\sigma(G^{(m)})$ is given by

(a) $\lambda_{z,1} = X_1^z + \frac{\mu_i}{3}, \ldots, \lambda_{z,m} = \sum_{j=0}^{m-1}(\frac{1}{3})^j X_{m-j}^z + (\frac{1}{3})^{m-1}(\frac{\mu_i}{3}) \in \sigma(G^{(m)})$ *with multiplicity* 1 *for each of them,*

(b) $0 \in \sigma(G^{(1)})$ *with multiplicity* $k(k-2)(k+1)^{(m-1)}$.

where, $\lambda_{z,j}$ *are the eigenvalues for* $G^{(m)}$ *such that* j *represents* j^{th} *corona product with* $z = 1, 2, 3$ *for the* 3 *angles i.e.* $\frac{\theta_j}{3}, \frac{2\pi+\theta_j}{3}, \frac{4\pi+\theta_j}{3}$ *as shown in following sub-expressions*

$$X_l^z = \frac{2}{3}\cos\frac{y\pi+\theta_l}{3}\sqrt{(\sum_{j=1}^{l-1}(\frac{1}{3})^{j-1}X_{l-j}^z + (\frac{1}{3})^{l-1}\mu_i)^2 + (6k-3)},$$

$$X_1^z = \frac{2}{3}\cos\frac{y\pi+\theta_1}{3}\sqrt{\mu_i^2 + (6k-3)}$$

$$\theta_m = \cos^{-1}\left(\frac{2(\sum_{j=0}^{m-1}(\frac{1}{3})^j X_{m-j}^z + (\frac{1}{3})^{m-1}(\frac{\mu_i}{3}))^3 - (9k-18)(\sum_{j=0}^{m-1}(\frac{1}{3})^j X_{m-j}^z + (\frac{1}{3})^{m-1}(\frac{\mu_i}{3})) + (54k-54)}{2((\sum_{j=0}^{m-1}(\frac{1}{3})^j X_{m-j}^z + (\frac{1}{3})^{m-1}(\frac{\mu_i}{3}))^2 + (6k-3))^{\frac{3}{2}}}\right) - y\pi,$$

where m *is the* m^{th} *corona product and* $y = 0, 2, 4$ *for the three angles.*

Here, $\lambda_{z,j} \in \left[-\mu_i + \frac{2\mu_i}{3^j} - \frac{2j\sqrt{(6k-3)}}{3}, \mu_i + \frac{2j\sqrt{(6k-3)}}{3}\right]$, *where* $j = 1, \ldots, m$.

Laplacian spectra

The Laplacian matrix $L(G^{(m)})$ associated with $G^{(m)}$ has the form

$$\mathbf{L}(\mathbf{G}^{(\mathbf{m})}) = \begin{bmatrix} \mathbf{L}(\mathbf{G}^{(\mathbf{m-1})}) + n\mathbf{I}_{n(n+1)^{m-1}} & -\mathbf{1}^T_{n(n+1)^{m-1}} \otimes \mathbf{I}_{\mathbf{n(n+1)^{m-1}}} \\ -\mathbf{1}_{n(n+1)^{m-1}} \otimes \mathbf{I}_{\mathbf{n(n+1)^{m-1}}} & (\mathbf{L}(\mathbf{G}) + \mathbf{I}_n) \otimes \mathbf{I}_{\mathbf{n(n+1)^{m-1}}} \end{bmatrix}$$

where $L(G^{(m-1)})$ is the Laplacian matrix of $G^{(m-1)}$, I_n and $I_{n(n+1)^{m-1}}$ are the identity matrices. We denote the Laplacian spectra $L(G^{(m)})$ of $G^{(m)}$ by

$$S(G^{(m)}) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n(n+1)^{(m)}}\} \tag{6}$$

where $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n(n+1)^{(m)}}$. The second smallest eigenvalue (λ_2) of $L(G)$ is termed as the algebraic connectivity of a graph.

- Application of algebraic connectivity
  - Indicates the robustness against disconnecting the network.

Laplacian spectra

In the following theorem, we determine the elements of $S(G^{(m)})$ in terms of the Laplacian eigenvalues of the basic graph, $G^{(0)} = G$ where $S(G) = \{0 = \nu_1, \nu_2, \ldots, \nu_n\}$ (where, $\nu_1 \leq \nu_2 \leq \ldots \leq \nu_n$) is the Laplacian spectra of G.

Theorem

Let G be a simple connected graph. We denote the algebraic connectivity of G and $G^{(m)}$ by $a(\nu_2)$ and $a(\lambda_2)$ respectively. The, Laplacian spectra $S(G^{(m)})$ of G is given by

(a) $\dfrac{\nu_i + (n+1)\sum_{i=0}^{m-1} 2^i \pm \sum_{i=1}^{m} z_i}{2^m} \in S(G^{(m)})$ *with multiplicity* 1 *for* $i = 1, \ldots, n(n+1)^{m-1}$. *where,*
$z_1 = \sqrt{(\nu_i + n + 1)^2 - 4\nu_i}, \ldots,$
$z_m = \sqrt{(\nu_i + (n+1)\sum_{i=0}^{m-1} 2^i \pm \sum_{i=1}^{m-1} z_i)^2 - 2^{(m+1)}(\nu_i + (n+1)\sum_{i=0}^{m-2} 2^i \pm \sum_{i=1}^{m-1} z_i)}$

(b) $\nu_i + 1 \in S(G^{(m)})$ *with multiplicity* $n(n+1)^{m-1}$ *for* $i = 2, \ldots, n$.

Hence, the algebraic connectivity of $S(G^{(m)})$ is

$$a(\lambda_2) = \frac{\nu_2 + (n+1)\sum_{i=0}^{m-1} 2^i - \sum_{i=1}^{m} z_i}{2^m} < 1$$

where z_i can be defined as above.

Signless Laplacian spectra

The signless Laplacian matrix $S_Q(G^{(m)})$ of $G^{(m)}$ is of the form

$$\mathbf{S_Q(G^{(m)})} = \begin{bmatrix} \mathbf{S_Q(G^{(m-1)})} + n\mathbf{I}_{n(n+1)^{m-1}} & \mathbf{1}^T_{n(n+1)^{m-1}} \otimes \mathbf{I_{n(n+1)^{m-1}}} \\ \mathbf{1}_{n(n+1)^{m-1}} \otimes \mathbf{I_{n(n+1)^{m-1}}} & (\mathbf{S_Q(G)} + \mathbf{I}_n) \otimes \mathbf{I_{n(n+1)^{m-1}}} \end{bmatrix}$$

where $S_Q(G^{(m-1)})$ is the signless Laplacian matrix of $G^{(m-1)}$, I_n and $I_{n(n+1)^{m-1}}$ are the identity matrices. The spectrum of signless Laplacian of $Q(G^{(m)})$ for $G^{(m)}$ is denoted by

$$S_Q(G^{(m)}) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n(n+1)^{(m)}}\} \tag{7}$$

where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n(n+1)^m}$. We will define the $S_Q(G^{(m)})$ inspired by the work of Cui et al. on two graphs[1] and here taking the basic initial graph G as the r-regular graph.

- Application of signless Laplacian spectra
  - Used for determining the dynamical behaviour of networks.
  - Used for determining the clusters in the data[2].

[1] Cui, Shu-Yu and Tian, Gui-Xian.: The spectrum and the signless Laplacian spectrum of coronae. Linear Algebra Appl. 437, 1692–1703 (2012).

[2] Lucińska, Małgorzata, and Sławomir T. Wierzchoń. Spectral clustering based on k-nearest neighbour graph. Computer Information Systems and Industrial Management. Springer Berlin Heidelberg, 254–265, (2012).

Signless Laplacian spectra

In the following theorem, we derived the elements of $S_Q(G^{(i)})$ in terms of the signless Laplacian eigenvalues of a *regular* basic graph, $G^{(0)} = G$ such that $S_Q(G) = (q_1, q_2, \ldots, q_n = 2r)$ (where, $q_1 \leq q_2 \leq \ldots \leq q_n = 2r$).

Theorem

Let G be a simple connected graph. Then, $S_Q(G^{(m)})$ is given by

(a) $\lambda_i = \dfrac{q_i + n\sum_{i=0}^{m-1} 2^i + r\sum_{i=1}^{m} 2^i + \sum_{i=0}^{m-1} 2^i \pm \sum_{j=1}^{m} z_j}{2^m}$ *with multiplicity of* 1, *for* $i = 1, \ldots, n(n+1)^{m-1}$
where,
$z_j = \sqrt{(q_i + n\sum_{i=0}^{j-1} 2^i + r(\sum_{i=1}^{j-1} 2^i - 2^j) + (\sum_{i=0}^{j-2} 2^i - 2^{j-1}) \pm \sum_{i=1}^{j-1} z_i)^2 + 2^{2j}.n}$ *for* $j = 2, \ldots, m$ *and*
$z_1 = \sqrt{((q_i + n) - (2r+1))^2 + 4n}.$

(b) $q_j + 1$ *with the multiplicity of* $n(n+1)^{m-1}$ *for* $j = 1, \ldots, n-1$

Hence, spectral radius of $S_Q(G^{(m)})$ *is*

$$q(S_Q(G^{(m)})) = \frac{q_i + n\sum_{i=0}^{m-1} 2^i + r\sum_{i=1}^{m} 2^i + \sum_{i=0}^{m-1} 2^i + \sum_{j=1}^{m} z_j}{2^m}$$

where z_j *is defined as above.*

Signless Laplacian spectra for star graphs

Consider star graph S_k which is an irregular graph. In the theorem below, we determine explicit formula of signless Laplacian elements of $G^{(m)} = S_k^{(m)}$.

Theorem

Let $S_Q(G) = \{q_1, q_2, \ldots, q_n\}$ with $q_1 \leq q_2 \leq \ldots \leq q_n$. Then, $S_Q(G^{(1)})$ is given by

(a) $\lambda_z = X_1^z + \frac{q_i+2k+2}{3} \in \sigma(G^{(1)})$ *with multiplicity* 1, *and*

(b) $q_j + 1 \in \sigma(G^{(1)})$ *where for* $j = 2, \ldots, n-1$ *with multiplicity of each of them as* k

where, λ_z are the eigenvalues for $G^{(1)}$ with $z = 1, 2, 3$ for the 3 angles i.e. $\frac{\theta}{3}, \frac{2\pi+\theta}{3}, \frac{4\pi+\theta}{3}$ as shown in following sub-expressions

$$X^z = \frac{2}{3}\cos\frac{y\pi+\theta}{3}\sqrt{q_i^2 + q_i(k-2) + (k+1)^2},$$

$$\theta = \cos^{-1}\left(\frac{2q_i^3+(3k-6)q_i^2-3(k^2-k-2)q_i+(70k-94-12\sum_{a=1}^{k-2}(a+2)(k-a-1)}{2(q_i^2+q_i(k-2)+(k+1)^2)^{\frac{3}{2}}}\right) - y\pi$$

where $y = 0, 2, 4$. Here, $\lambda_z \in \left[-q_i + \frac{2q_i}{3} + \frac{Y-(A+\sqrt{4B-A^2})}{3}, q_i + \frac{Y+A+\sqrt{4B-A^2}}{3}\right]$

where $A = (k-2)$, $B = (k+1)^2$, $Y = (2k+2)$.

Signless Laplacian spectra for star graphs

Corollary

Let $S_Q(G)$ for star graph S_k for each $k \geq 3$ is $\{q_1, q_2, \ldots, q_n\}$ with $q_1 \leq q_2 \leq \ldots \leq q_n$. Then, $S_Q(G^{(m)})$ is given by

(a) $\lambda_{z,1} = X_1^z + \frac{q_i+2k+2}{3}, \ldots, \lambda_{z,m} = \sum_{j=0}^{m-1}(\frac{1}{3})^j X_{m-j} + (\frac{1}{3})^m q_i + (2k+2)\sum_{j=1}^{m}(\frac{1}{3})^j \in \sigma(G^{(m)})$ *with multiplicity* 1, *and*

(b) $q_i + 1 \in \sigma(G^{(1)})$ *where for* $i = 2, \ldots, n-1$ *with multiplicity of each of them as* $k(k+1)^{(m-1)}$.

where, $\lambda_{z,j}$ *are the eigenvalues for* $G^{(1)}$ *(for* j^{th} *corona product) with* $z = 1, 2, 3$ *for the* 3 *angles i.e.* $\frac{\theta_j}{3}, \frac{2\pi+\theta_j}{3}, \frac{4\pi+\theta_j}{3}$ *as shown in following sub-expressions*

$X_l^z = \sqrt{(A)^2 + A + (k+1)^2}$

$X_1^z = \frac{2}{3}\cos\frac{y\pi+\theta}{3}\sqrt{q_i^2 + q_i(k-2) + (k+1)^2},$

where $A = (\sum_{j=0}^{m-2}(\frac{1}{3})^j X_{m-j-1} + (\frac{1}{3})^{m-1} q_i + (2k+2)\sum_{j=1}^{m-1}(\frac{1}{3})^j)$

$$\theta = \cos^{-1}\left(\frac{2\lambda_{z,m-1}^3 + (3k-6)\lambda_{z,m-1}^2 - 3(k^2-k-2)\lambda_{z,m-1} + (70k-94-12\sum_{a=1}^{k-2}(a+2)(k-a-1)}{2(\lambda_{z,m-1}^2 + \lambda_{z,m-1}(k-2) + (k+1)^2)^{\frac{3}{2}}}\right) - y\pi$$

where $y = 0, 2, 4$ *and* $\lambda_{z,m-1}$ *is as defined in part(a) of corollary. Here,*

$$\lambda_{z,1} \in \left[-q_i + \frac{2q_i}{3} + \frac{Y-(A+\sqrt{4B-A^2})}{3}, q_i + \frac{Y+A+\sqrt{4B-A^2}}{3}\right], \ldots,$$

$$\lambda_{z,m} \in \left[-q_i + \frac{2q_i}{3m} - \frac{m(A+\sqrt{4B-A^2})}{3} + Y(-2\sum_{i=1}^{m}\frac{m-i}{3^i+1} + \sum_{i=1}^{m}3^{-i}), q_i + m(\frac{Y+A+\sqrt{4B-A^2}}{3})\right]$$

where $A = (k-2)$, $B = (k+1)^2$, $Y = (2k+2)$.

Outline

Conclusions

- Proposed a model for generating complex networks based on corona product of the graphs.
- Investigated spectra, Laplacian spectra and signless Laplacian spectra of corona graphs.
- Derived the spectra and signless Laplacian spectra of corona graphs when the basic graph is a star graph.
- Corona graphs can be used as models for investigating the duplication mechanism of an individual gene for the formation of new proteins.

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Thanks

for

Listening