Bounds for the b-chromatic number of induced subgraphs and $G - e$

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Outline

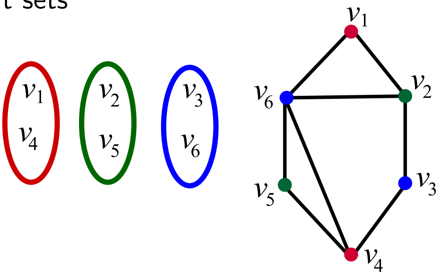
- Preliminaries
- Bounds for the b-chromatic number of induced subgraphs
- Bounds for $b(G-e)$ in terms of $b(G)$
- Extremal graphs
- References

b-coloring

1. The b-chromatic number was introduced by R.W. Irving andD.F. Manlove.
2. b-chromatic number has received wide attention from the time of its introduction.
3. They have shown that the determination of b(G) is NP-hard for general graphs, but polynomial for trees. Here is a motivation as to why we study b-coloring.

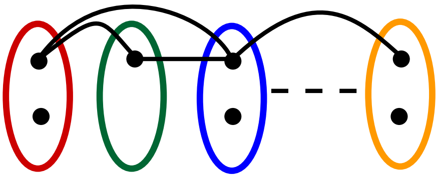
Motivation

1. All graphs considered are simple, undirected and finite.
2. A proper k-coloring is a map $f : V \to S$, where S is a set of distinct colors such that adjacent vertices receive distinct colors. Equivalently,
3. A proper k-coloring is partitioning of the vertex set into k independent sets



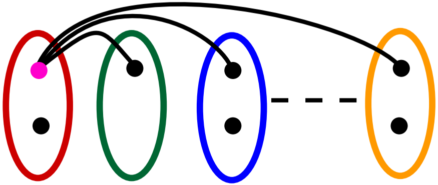
Motivation

1. **Chromatic number:** The chromatic number $\chi(G)$ of a graph G is the minimum number of colors needed for a proper vertex coloring of G.
2. For any chromatic coloring between any two classes there is an edge.
3. The minimum with this property – chromatic number
4. The maximum with this property – achromatic number $\psi(G)$.



Motivation

1. **Chromatic number:** The chromatic number $\chi(G)$ of a graph G is the minimum number of colors needed for a proper vertex coloring of G.
2. For any chromatic coloring every color class contains a color dominating vertex (c.d.v.).
3. The minimum with this property — chromatic number



Preliminaries

b-coloring of a graph

A b-coloring of a graph G with k colors is a proper coloring of G using k colors in which each color class contains a color dominating vertex, that is, a vertex which has a neighbor in each of the other color classes.
The largest positive integer k for which G has a b-coloring using k colors is the b-chromatic number $b(G)$ of G.

$$\chi(G) \leq b(G) \leq \psi(G) \leq \Delta(G) + 1$$

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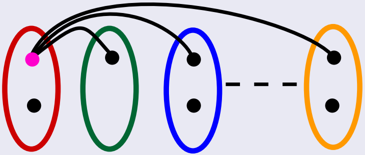

Preliminaries

m-degree of a graph

The vertices $x_1, x_2, ..., x_n$ of G are ordered such that $d(x_1) \geq d(x_2) \geq ... \geq d(x_n)$ then m-degree of a graph $m(G) = max\{1 \leq i \leq n : d(x_i) \geq i - 1\}$.

- $b(G) \leq m(G) \leq \Delta(G) + 1$

Lemma (J. Kratochvil, Z. Tuza, M. Voigt)

Let G be a non-trivial connected graph. Then $b(G) = 2$ if and only if G is bipartite and has a full vertex in each part of the bipartition.

Problem

- $\chi(G-v) = \chi(G)$ or $\chi(G)-1$.
  Also $\psi(G-v) = \psi(G)$ or $\psi(G)-1$.
- R. Balakrisnan, S. Francis Raj obtains the bounds for $b(G-v)$ in terms of $b(G)$.

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- Similarly $\chi(G-e) = \chi(G)$ or $\chi(G) - 1$.
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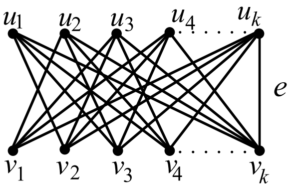

Figure: $b(G) = 2$ and $b(G-e) = k$

Problem

- The bounds for $b(G-e)$.
- The bounds for $b(G-S)$ where $G-S$ is an induced subgraph of G.
- The extremal graphs which attains the upper bound for $b(G-e)$.

Upper bound for $b(G-S)$

Theorem

For any connected graph G with $n \geq 5$ vertices and for any $S \subset V(G)$,

$$b(G-S) \leq b(G) + \left\lceil \frac{n-|S|}{2} \right\rceil - 2, \text{ for all } 1 \leq |S| \leq n-3.$$

Upper bound for $b(G-S)$

Proof

$$b(G-S) = b(G) + \left\lceil \frac{n-|S|}{2} \right\rceil - 2 + k, (k \geq 1) \qquad (1)$$

- Let c' be a b-chromatic coloring of $G-S$.
  P' – singleton classes of c'
  Q' – remaining classes of c'.

Upper bound for $b(G-S)$

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- Let c' be a b-chromatic coloring of $G-S$.
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- $(1) \Rightarrow b(G-S) - b(G) = \left\lceil \frac{n-|S|}{2} \right\rceil - 2 + k \geq 1.$

Upper bound for $b(G-S)$

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- Vertices of P' form a clique, $|P'| \leq b(G) - 1$.

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- $(1) \Rightarrow b(G-S) - b(G) = \left\lceil \frac{n-|S|}{2} \right\rceil - 2 + k \geq 1$.
- Vertices of P' form a clique, $|P'| \leq b(G) - 1$.
- Also $b(G-S) = |P'| + |Q'|$, we get $|Q'| \geq \left\lceil \frac{n-|S|}{2} \right\rceil - 1 + k$.

Upper bound for $b(G-S)$

Case (1) Both n and $|S|$ are of same parity

- $|Q'| \geq \frac{n-|S|}{2} - 1 + k \geq \frac{n-|S|}{2}$, and $|V(Q')| \geq n - |S|$.

Upper bound for $b(G-S)$

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- $|Q'| \geq \frac{n-|S|}{2} - 1 + k \geq \frac{n-|S|}{2}$, and $|V(Q')| \geq n - |S|$.
- Also $|V(Q')| \leq n - |S|$. Hence $|V(Q')| = n - |S|$, $|P'| = 0$, $|Q'| = \frac{n-|S|}{2}$ and $b(G-S) = \frac{n-|S|}{2}$.

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- $(1) \Rightarrow b(G-S) \geq 2 + \frac{n-|S|}{2} - 2 + 1 = \frac{n-|S|}{2} + 1$.

Upper bound for $b(G-S)$

Case (1) Both n and $|S|$ are of same parity

- $|Q'| \geq \frac{n-|S|}{2} - 1 + k \geq \frac{n-|S|}{2}$, and $|V(Q')| \geq n - |S|$.
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- $(1) \Rightarrow b(G-S) \geq 2 + \frac{n-|S|}{2} - 2 + 1 = \frac{n-|S|}{2} + 1$.

Case (2) Both n and $|S|$ are of different parity

- $|Q'| \geq \frac{n-|S|+1}{2} - 1 + k \geq \frac{n-|S|+1}{2}$.
- $|V(Q')| \geq n - |S| + 1$.

Lower bound for $b(G-S)$

Theorem

For any connected graph G with $n \geq 5$ vertices and for any $S \subset V(G)$,

$$b(G-S) \geq b(G) - \left\lfloor \frac{n+|S|}{2} \right\rfloor + 2, \text{ for all } 1 \leq |S| \leq n-4.$$

Corollary

R. Balakrisnan, S. Francis Raj (Bounds for the b-chromatic number of $G - v$)

For any connected graph G with $n \geq 5$ vertices and for any $v \in V(G)$,

$$b(G) - \left(\left\lceil \frac{n}{2} \right\rceil - 2\right) \leq b(G - v) \leq b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

- By substituting $|S| = 1$ in previous Theorem.

Extremal graph

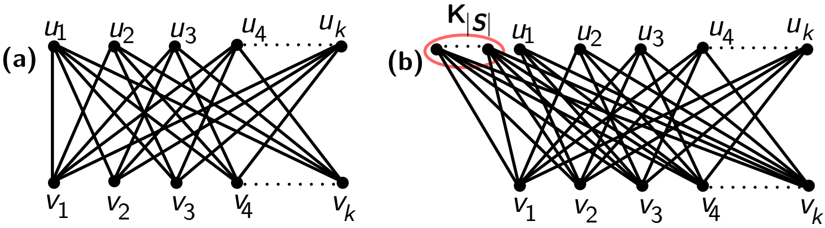

Figure: Graphs which attain the bounds.

Bounds for $b(G-e)$ in terms of $b(G)$

Theorem

For any connected graph G with n vertices and for any $e \in E(G)$,

$$b(G) - 1 \leq b(G-e) \leq b(G) + \left\lceil \frac{n}{2} \right\rceil - 2.$$

Bounds for $b(G - e)$ in terms of $b(G)$

Theorem

For any connected graph G with n vertices and for any $e \in E(G)$,

$$b(G) - 1 \leq b(G - e) \leq b(G) + \left\lceil \frac{n}{2} \right\rceil - 2.$$

Proof.

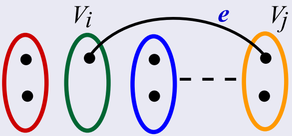

Figure: Lower bound for $b(G - e)$

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Extremal graphs

- $b(G - e) = b(G) + \lceil \frac{n}{2} \rceil - 2$

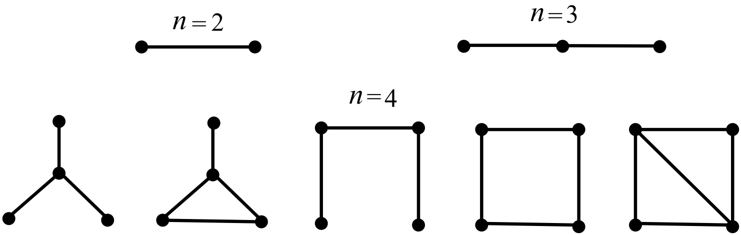

Figure: Extremal graphs when $n = 2, 3, 4$.

Extremal graphs

$$b(G) = b(G-e) - \left\lceil \frac{n}{2} \right\rceil + 2 \qquad (2)$$

- Let c' be a b-chromatic coloring of $G-e$, where $e = uv$.
  S' – singleton classes of c'
  T' – remaining classes of c'.

Extremal graphs

$$b(G) = b(G - e) - \left\lceil \frac{n}{2} \right\rceil + 2 \qquad (2)$$

- Let c' be a b-chromatic coloring of $G - e$, where $e = uv$.
  S' – singleton classes of c'
  T' – remaining classes of c'.
- As $n \geq 5$, $b(G - e) - b(G) \geq 1$.
- u and v belongs to the same class of c'.

Extremal graphs

$$b(G) = b(G-e) - \left\lceil \frac{n}{2} \right\rceil + 2 \quad (2)$$

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- As $n \geq 5$, $b(G-e) - b(G) \geq 1$.
- u and v belongs to the same class of c'.
- Vertices of S' form a clique, $|S'| \leq b(G) - 1$ and therefore $|T'| \geq \lceil \frac{n}{2} \rceil - 1$.

Extremal graphs

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Case (i) n is even.
$|T'| \geq \frac{n}{2} - 1$, and thus $|V(T')| \geq n - 2$.

Extremal graphs (n is even)

Here $b(G) = 3$.

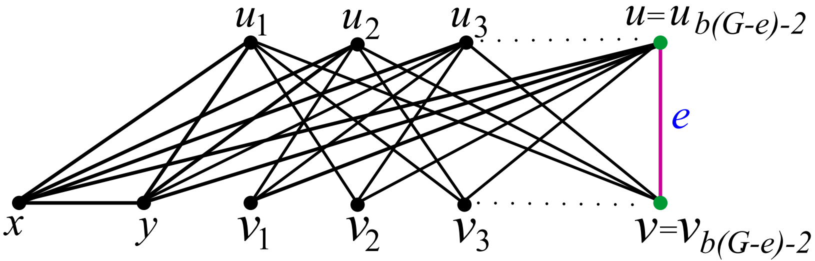


Figure: n is even and $|V(T')| = n - 2$.

Extremal graphs (n is even)

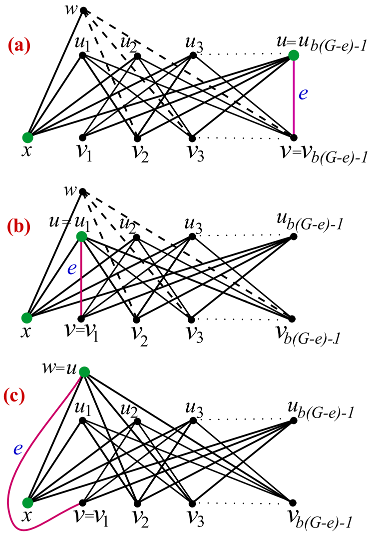


Figure: n is even, $|V(T')| = n - 1$ and x is adjacent to w.

Extremal graphs （n is even）

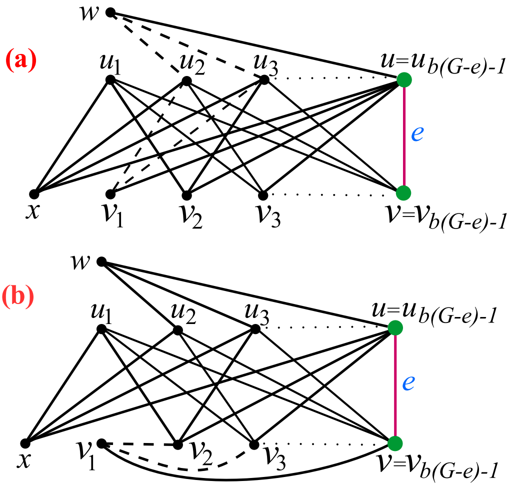


Figure: n is even, $|V(T')| = n-1$, x is non-adjacent to w and v_1.

Extremal graphs (n is even)

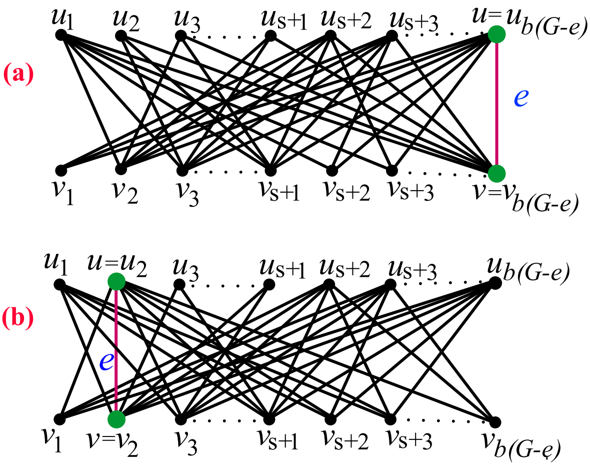
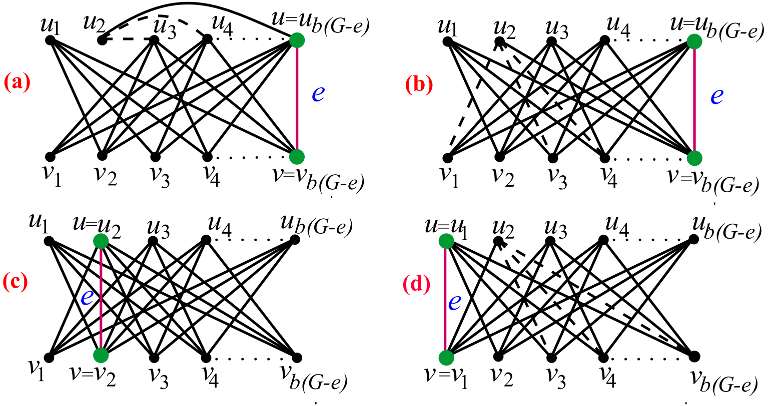


Figure: $|S'| = 0$, u_1 has one neighbor in each class and $s \geq 2$.

Extremal graphs (n is even)



u_1 has one neighbor in each class and $s=1$.

Figure: n is even, $b(G) = 2$ and $|V(T')| = n$.

Extremal graphs (n is even)

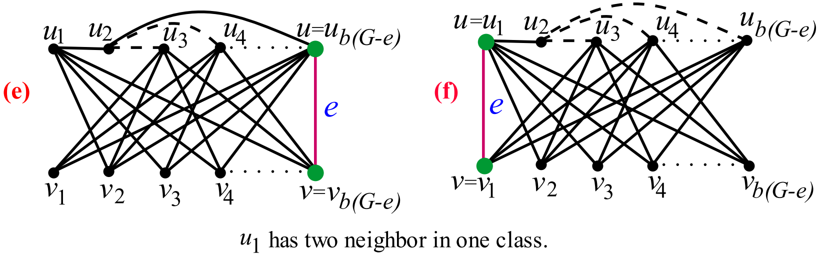


Figure: n is even, $b(G) = 2$ and $|V(T')| = n$.

Extremal graphs (n is odd)

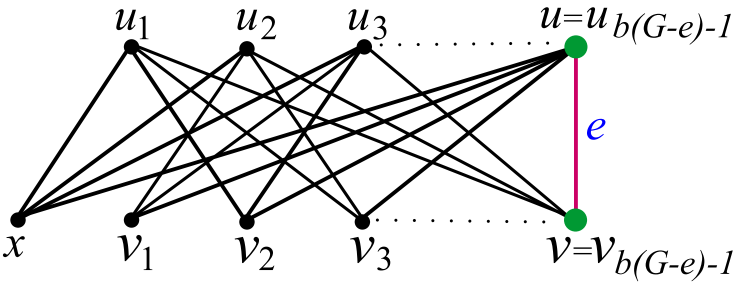


Figure: n is odd and $|V(T')| = n - 1$.

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Thank You