Obstruction Characterizations of Representable Graphs and Digraphs

Pavol Hell, SFU

2015 CALDAM

Interval graph

Vertices v can be represented by intervals I_v, so that

$$v \sim w \iff I_v \cap I_w \neq \emptyset$$

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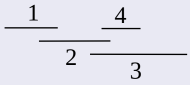
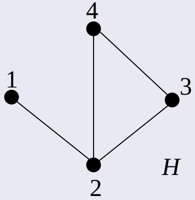
Example





Applications

Food webs, resource allocation, genetics, etc.

Benzer 1959, Cohen 1978, Klee 1969

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Algorithms

$O(m+n)$ recognition algorithms

Booth-Lueker 1976, Korte-Mohring 1989, Habib-McConnell-Paul-Viennot 1998, Corneil-Olariu-Stewart 1998

Greedy $O(n)$ optimization algorithms

Gavril 1974, Rose-Tarjan-Lueker 1976

Characterizations

- Ordering of the rows of the maxclique matrix

  Fulkerson-Gross 1965
- Via complements and comparability graphs

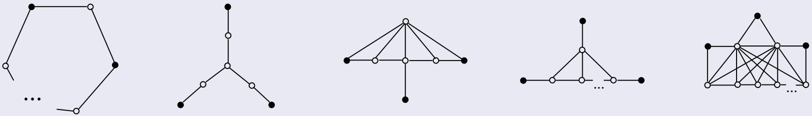
  Gilmore-Hoffman 1964
- Induced subgraph obstructions

  Lekkerkerker-Boland 1962

Interval Graphs

Lekkerkerker-Boland 1962

H is an interval graph $\iff$ H has no induced subgraph from



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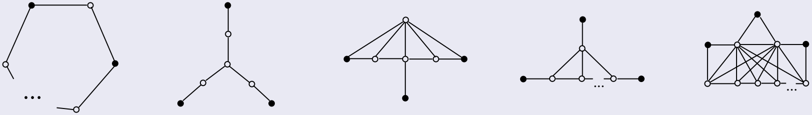

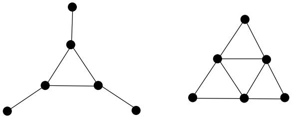

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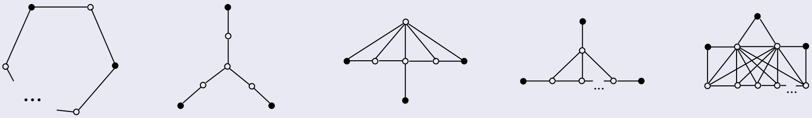

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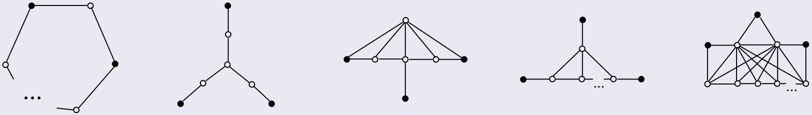
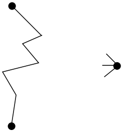


Asteroidal triple (AT)

Any two joined by a path avoiding the neighbours of the third



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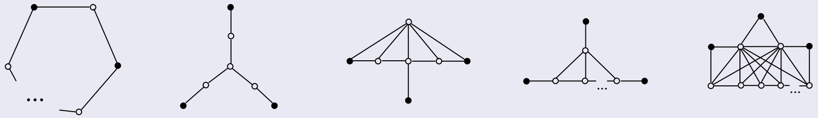

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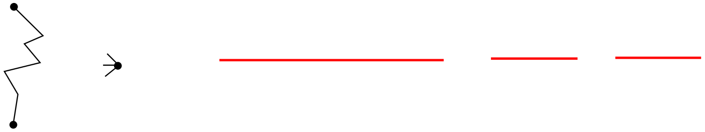

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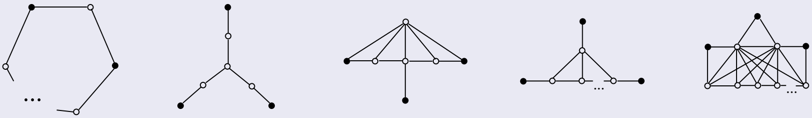

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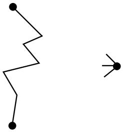

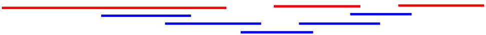

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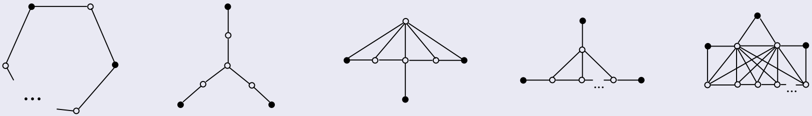
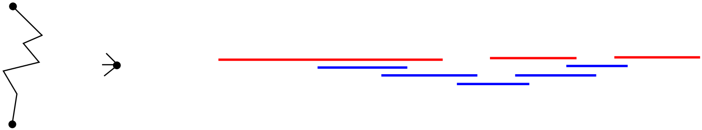
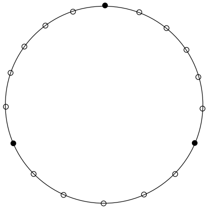


Lekkerkerker-Boland 1962

H is an interval graph $\iff$ H has no AT or induced $C_k, k \geq 4$.



A Structural Characterization



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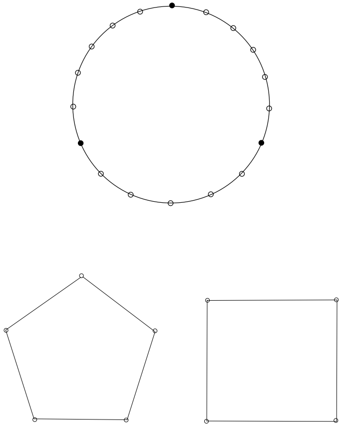

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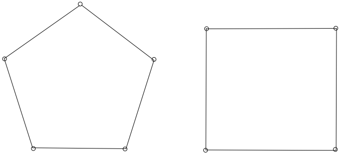

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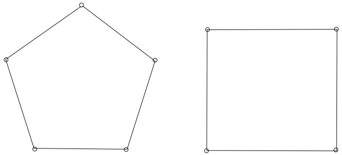

Kratsch-McConnell-Mehlhorn-Spinrad 2006

$O(m+n)$ certifying recognition algorithm

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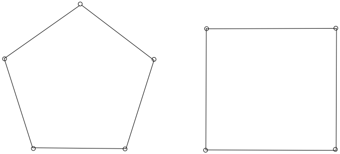

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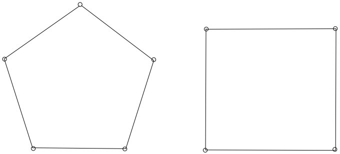

Can these obstructions be unified?

Is there a common explanation?

An Ordering Characterization

The umbrella property

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$$\iff$$

$V(H)$ can be linearly ordered by $<$ so that for $u < v < w$

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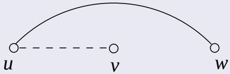

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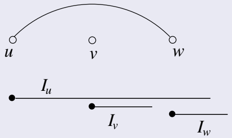

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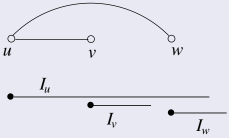

An Equivalent Ordering Characterization

Min-ordering

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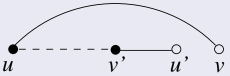

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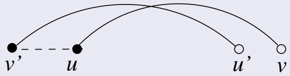

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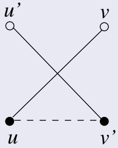
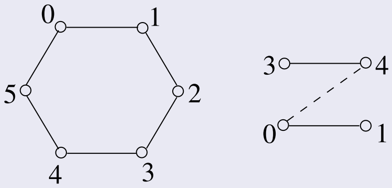


C_6 is not an interval graph



Another Obstruction

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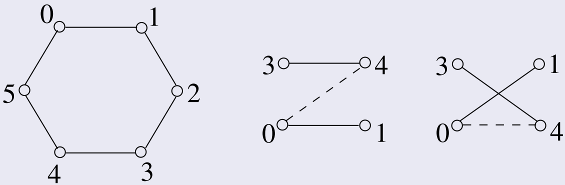

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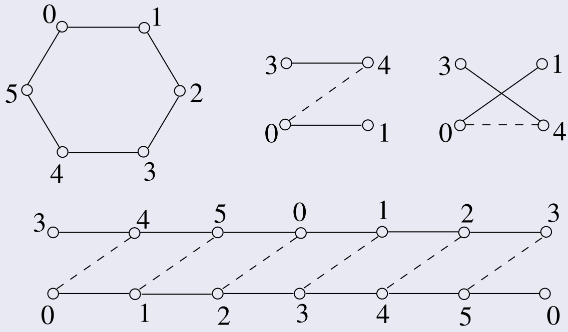
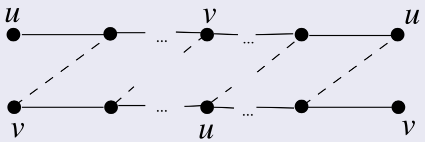


A Pulsed Obstruction

Invertible pair: a new obstruction



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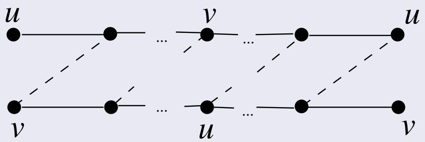

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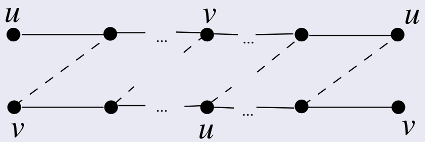

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Feder+H+Huang+Rafiey 2012

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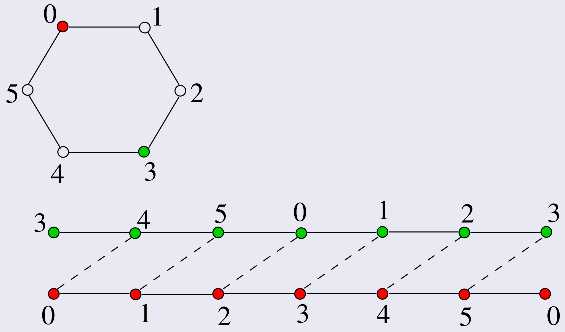

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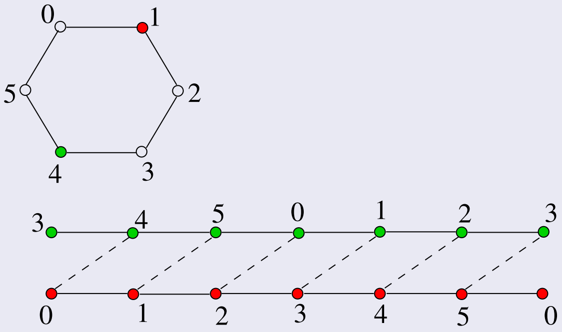

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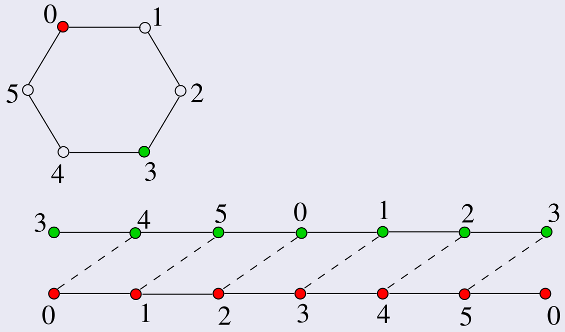

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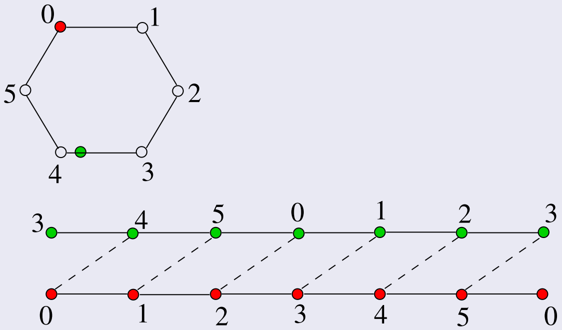

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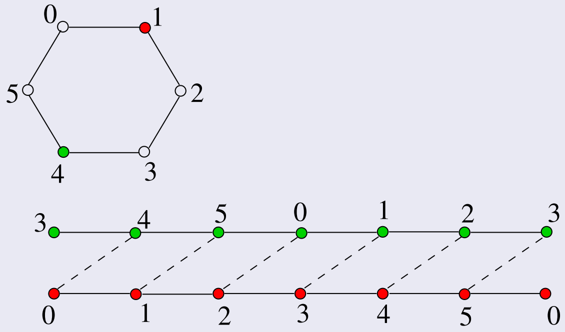

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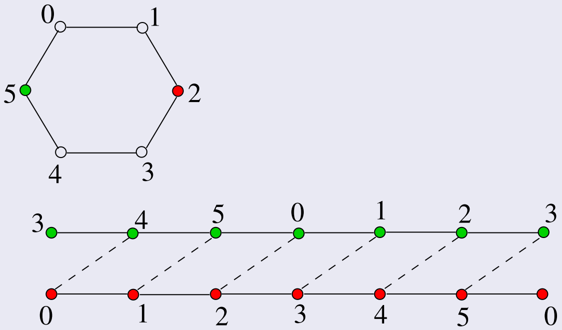

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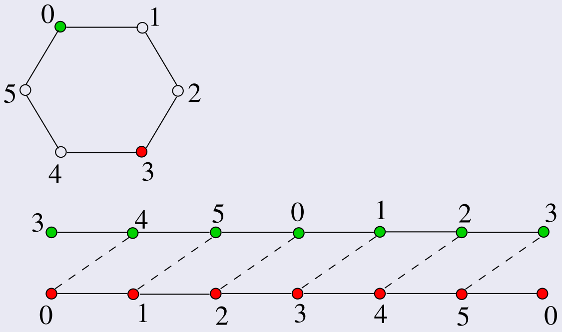

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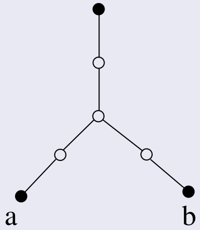
Each AT has an invertible pair



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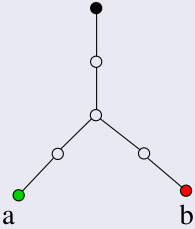
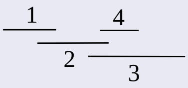
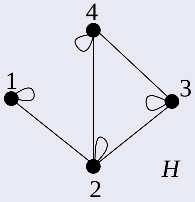


Observation

Interval graphs are reflexive (have all loops)





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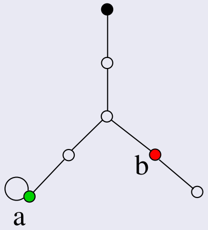

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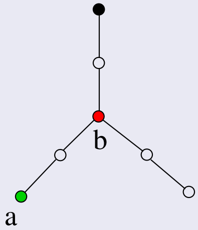

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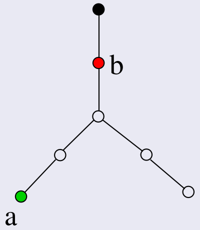

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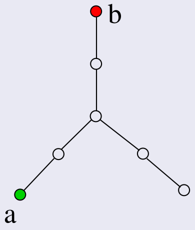

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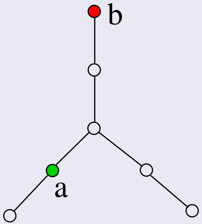

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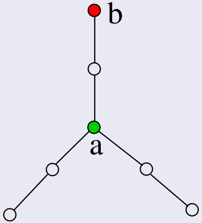

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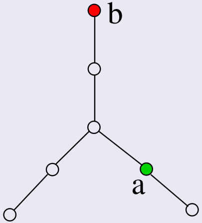

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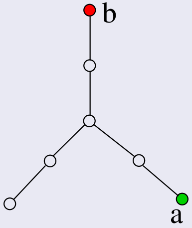

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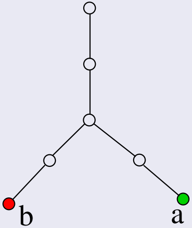

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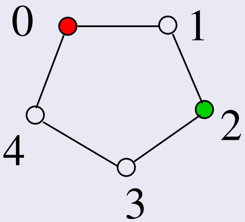

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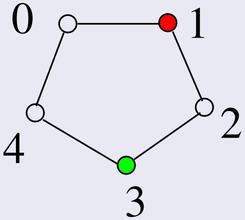

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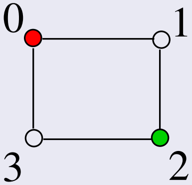

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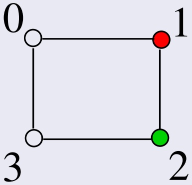

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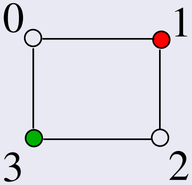

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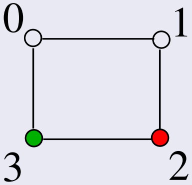

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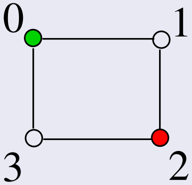
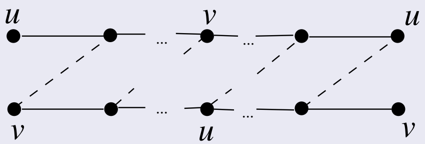


A pulsed obstruction characterization

H is an interval graph $\iff$ it has no invertible pair



Feder+H+Huang+Rafiey 2012

Qui a tué le Duc de Densmore?

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Party with ex-wives

- Anne met Felicia, Cynthia, Georgia, Emilie, and Betty
- Betty met Cynthia, Anne, and Helen
- Cynthia met Anne, Emily, Diane, Betty, and Helen
- Diane met Cynthia and Emily
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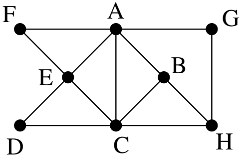

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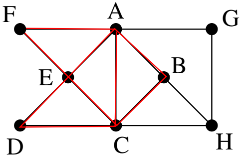

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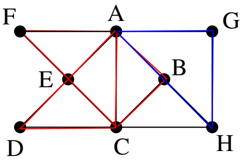

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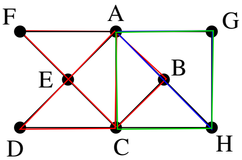

Interval Digraphs

An interval digraph

Vertices can be represented by pairs of intervals I_v, J_v, so that

$$v \to w \iff I_v \cap J_w \neq \emptyset$$

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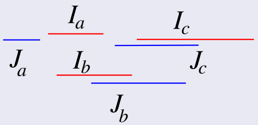

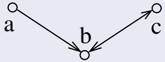

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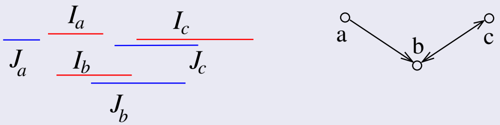

Sen-Das-Roy-West 1989

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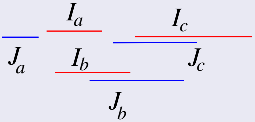

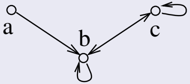

Sen-Das-Roy-West 1989

Characterizations?

Characterizations?

- Ordering of the rows and columns of the adjacency matrix
- Via complements and Ferrers digraphs

Sen-Das-Roy-West 1989

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Recognition in $O(n^2m^7)$

Mueller 1997

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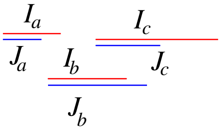
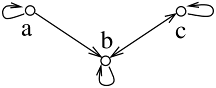
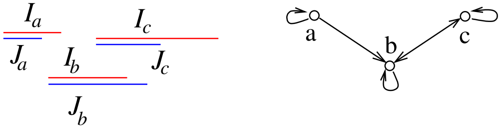
An obstruction characterization
A low-degree polynomial recognition algorithm ?
A polynomial certifying algorithm

Reflexive Digraphs

Reflexive Digraphs







An adjusted interval digraph

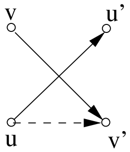
Vertices can be represented by pairs of *adjusted* intervals I_v, J_v

$$v \to w \iff I_v \cap J_w \neq \emptyset$$

Min-ordering

H is an adjusted interval digraph if and only if $V(H)$ can be linearly ordered by $<$ so that for $u < v$ and $u' > v'$

$$u \to u' \text{ and } v \to v' \implies u \to v'$$



Adjusted Interval Digraphs

H is an adjusted interval digraph $\iff$ it has no invertible pair

Feder+H+Huang+Rafiey 2012

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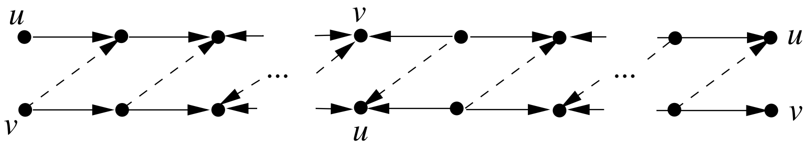

Adjusted Interval Digraphs

Recognition

A certifying $O(m^2)$ algorithm, via reachability in an auxiliary digraph

Feder+H+Huang+Rafiey 2012

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- Is there an $O(m+n)$ recognition algorithm?

Recognition

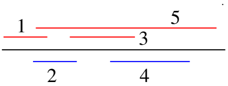
A certifying $O(m^2)$ algorithm, via reachability in an auxiliary digraph

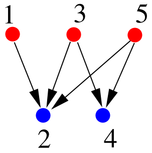
Feder+H+Huang+Rafiey 2012

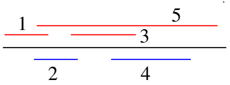
- Is there an $O(m+n)$ recognition algorithm?
- Are there interesting problems solvable in polynomial time on this digraph family?

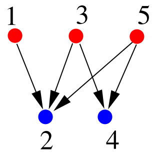
Irreflexive Digraphs

Irreflexive Digraphs









Special case

Vertices can be represented by pairs of intervals I_v, J_v where each v has either $I_v = \emptyset$ or $J_v = \emptyset$, and

$$v \rightarrow w \iff I_v \cap J_w \neq \emptyset$$

Interval Bigraphs

Bigraphs (with red and blue vertices)

Each edge joins a red vertex and a blue vertex

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Think of the edges as directed ($r \longrightarrow b$) but do not depict the arrows

Interval Bigraphs

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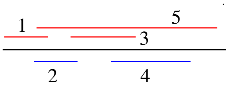
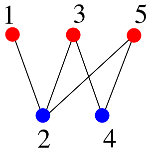
Interval bigraph

Representable by real intervals I_r, J_b (for r red and b blue)

$$r \sim b \Longleftrightarrow I_r \cap J_b \neq \emptyset$$





Interval Bigraphs

- Matrix characterization

  Sen-Das-Roy-West 1989

- $O(n^2 m^7)$ recognition

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Low-degree polynomial recognition algorithm ?
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What is the *right* bigraph analogue?

Bigraphs

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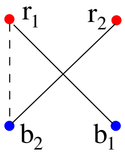

Two Directional Ray Graphs

A 2DR graph

Intersection graph of a family of UP and RIGHT rays

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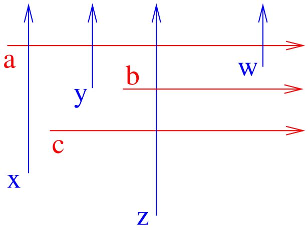

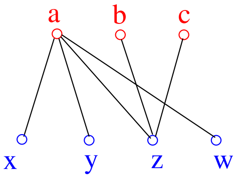

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- the associated poset of H has dimension two
- etc

H and Huang 2004; H and Rafiey 2011; Shrestha, Tayu, and Ueno 2010; Spinrad 1988

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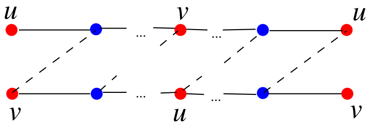

Two Directional Ray Graphs

Problems solvable in polynomial time on this class

- Weighted dominating set
- Induced matching
- Strong edge-colouring

Ershadi 2012, Takaoka, Tayu, Ueno 2013

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$O(n^2)$ recognition algorithm (via circular arc graphs McConnell 2003)
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H + Huang 2004

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H + Huang 2004

Corollary

Each interval bigraph is a 2DR graph

Background

Background

An H-colouring of G

$f : V(G) \to V(H)$ such that $u \to v$ in $G \implies f(u) \to f(v)$ in H

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$$u \to v \text{ and } u' \to v' \implies \min(u, u') \to \min(v, v')$$

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An H-colouring f of H^k with $f(x_1, x_2, \ldots, x_k) \in \{x_1, x_2, \ldots, x_k\}$

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Nice polymorphisms (commutative, associative, majority) are useful objects

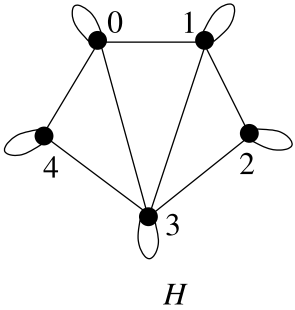
The List H-colouring problem

Given G with lists, can it be H-coloured by a mapping f that respects the lists, i.e., with each $f(v)$ is in the list of v?

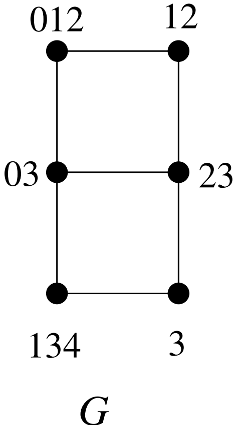
Background

Fixed graph H

Processors and connections



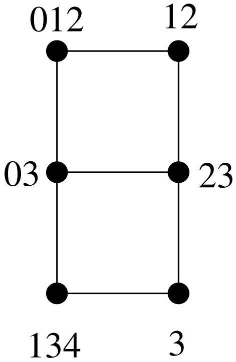
Background



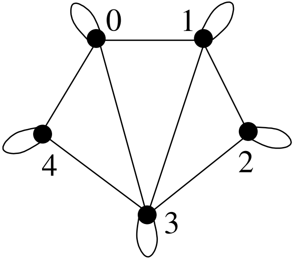
Input graph G with lists

Tasks and communications

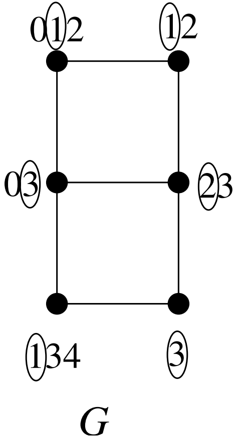
Background



G



H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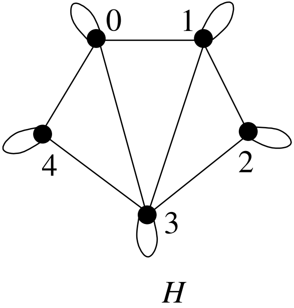
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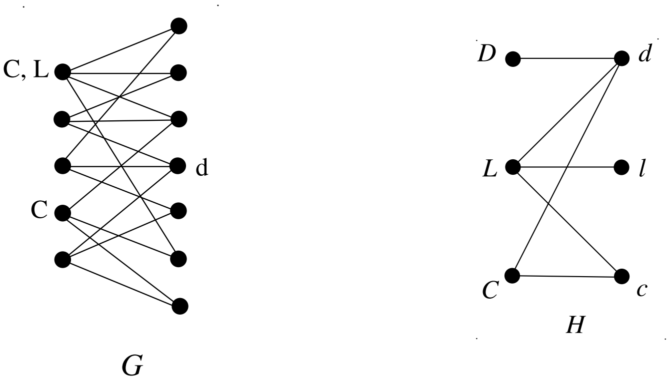
3
H

Another Example Application



Lists

Lists of allowed decisions

Background

Dichotomies for the List H-colouring problems

- If H is a reflexive graph then List H-colouring is polynomial time solvable if H is an **interval graph**, and is NP-complete otherwise
- If H is a reflexive digraph then List H-colouring is polynomial time solvable if H is an **adjusted interval digraph**, and is conjectured to be NP-complete otherwise
- If H is an irreflexive graph then List H-colouring is polynomial time solvable if H is a **2DR graph**, and is NP-complete otherwise

Feder, H 1995; Feder, H, Huang 1999, Feder, H, Huang, Rafiey 2012

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Feder, H 1995; Feder, H, Huang 1999, Feder, H, Huang, Rafiey 2012

If H has a nice polymorphism then List H-colouring is polynomial time solvable

Circular Arc Graphs

Circular arc graph

Vertices v can be represented by circular arcs I_v, so that

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Certifying algorithm?

New results

- A forbidden structure characterization

Circular Arc Graphs

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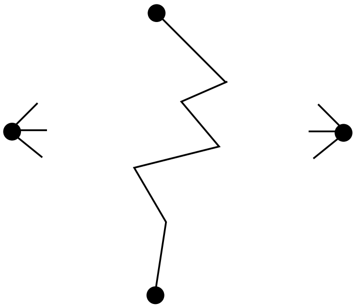
Francis-H-Stacho 2015

Blocking quadruple: an asteroid-like obstruction

Four vertices, between any two either a path avoiding the neighbours of both other vertices, or a path between the other two vertices avoiding the neighbours of our vertices.



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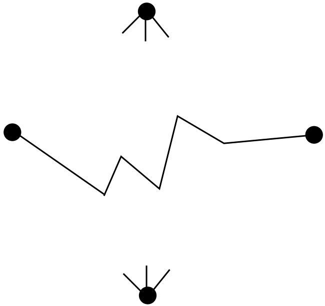

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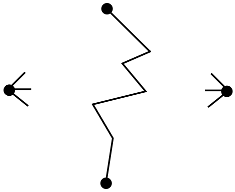

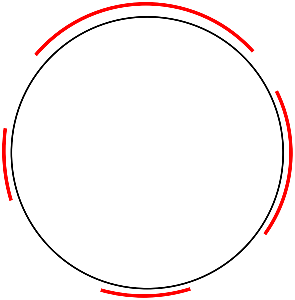

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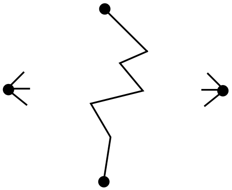

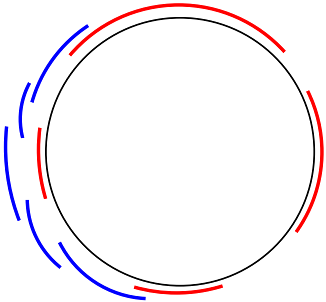

Francis-H-Stacho 2014

Circular Arc Graphs

Blocking quadruples (BQ) sometimes suffice

- A chordal K_5-free graph is a circular arc graph $\iff$ no BQ
- A chordal claw-free graph is a circular arc graph $\iff$ no BQ
- etc

Francis-H-Stacho 2014, Bonomo-Duran-Grippo-Safe 2009

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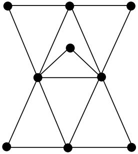
But not always

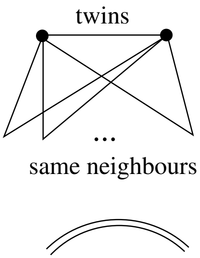


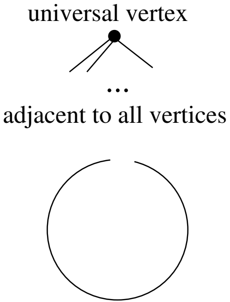
Our goal

Can we find something like invertible pair obstructions here?

The First Twist – Standard

H has no twins and universal vertices



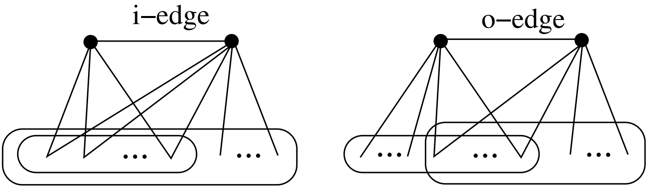


The First Twist – Standard

Each edge of H has a "type"

Type of edge uv

- Type i if $N[u] \subseteq N[v]$ ("inclusion")
- Type o if each u, v has a private neighbour ("overlap")



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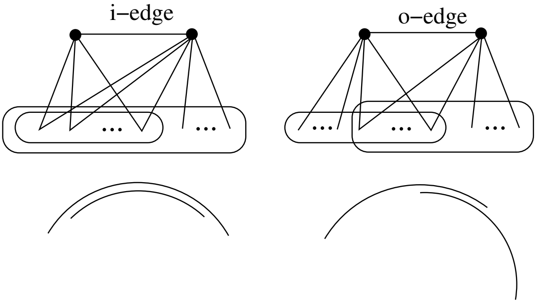

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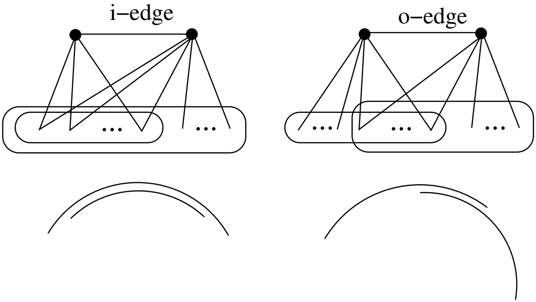


Hsu 1995

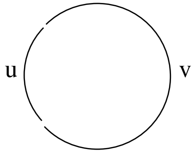
If H has a circular arc representation, then it has one corresponding to the labels

The Second Twist – New

Extend H to include "complements"

Circularly paired vertices u, v

- u and v are not adjacent
- $x \not\sim u \implies xv$ is an i-edge, and
  $x \not\sim v \implies xu$ is an i-edge



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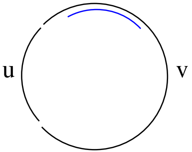

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If u is not circularly paired in H, we add a suitable new vertex $\overline{u}$ ($x \sim \overline{u} \iff xu$ is not an i-edge)

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Facts

Each H has a unique circular completion H^+
H is a circular arc graph $\iff H^+$ is a circular arc graph

Review all assumptions

- H has no twins and no universal vertices
- edges of H are labeled by their type i or o
- H is circularly complete

The Structural Characterization

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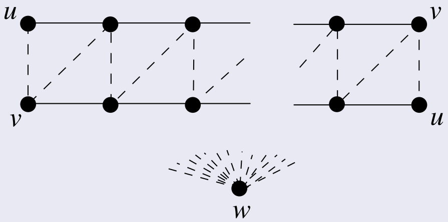
Anchored invertible pair



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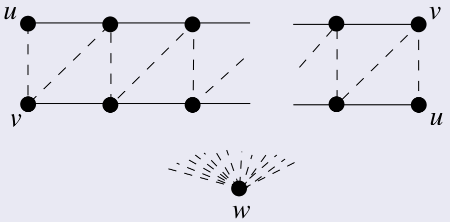

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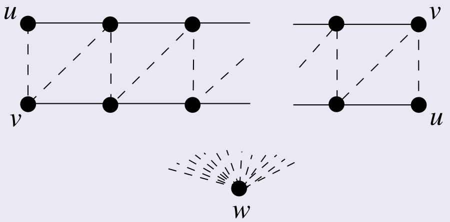

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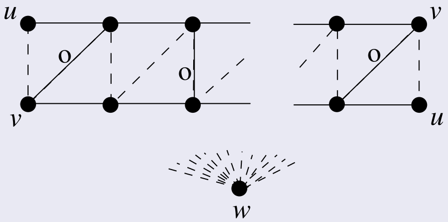

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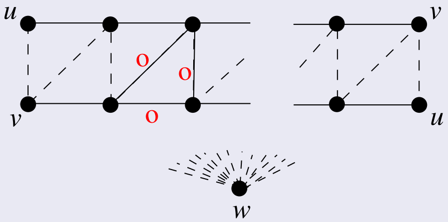

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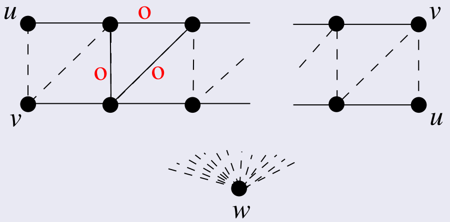

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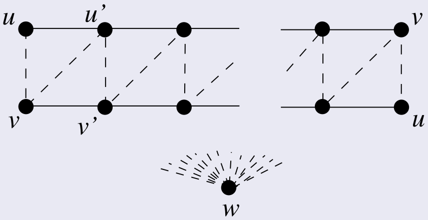

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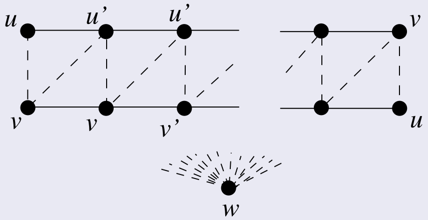

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Theorem

H is a circular arc graph $\iff$ it has no anchored invertible pair

Francis-H-Stacho 2015

Necessity

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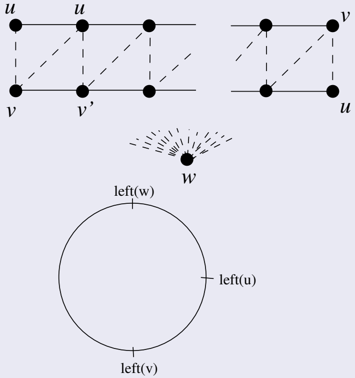

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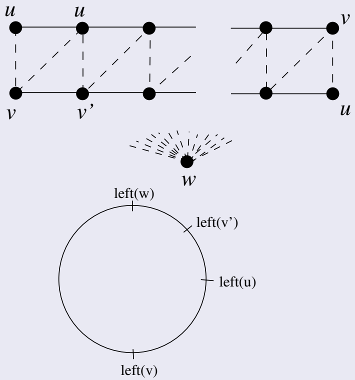

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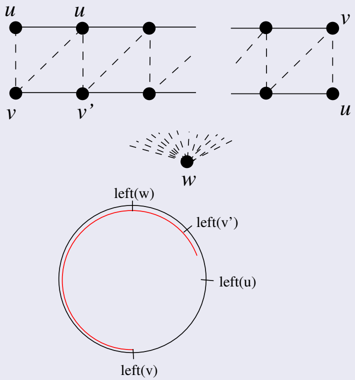

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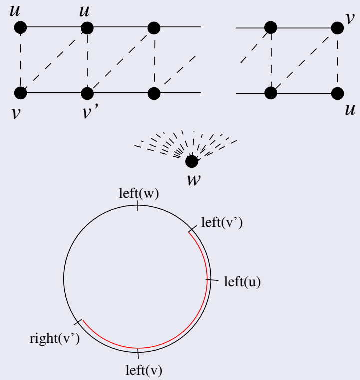

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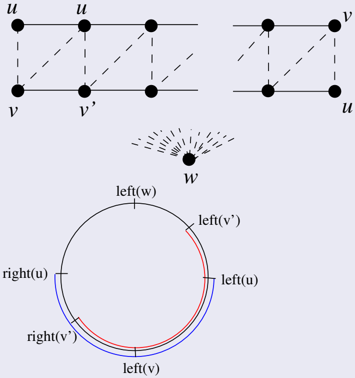

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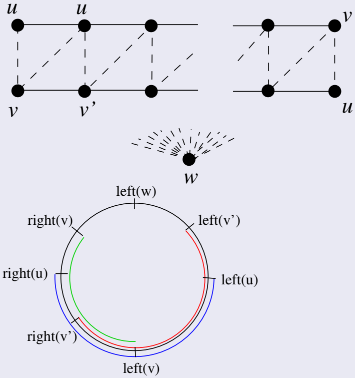

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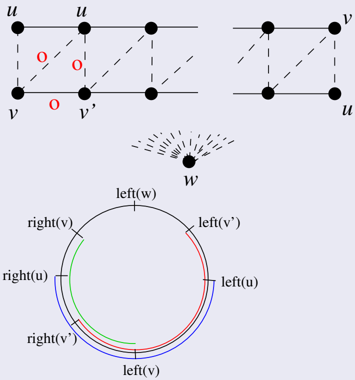

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A certifying recognition algorithm

All this can be done in time $O(n^3)$