# Separator Theorems for Interval Graphs and Proper Interval Graphs

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# Outline

1. Definition of Interval Graphs
2. Some Known Characterization of Interval Graphs
3. Separator Theorem for Interval Graphs
4. Forbidden Subgraph Characterization
5. Separator Theorem for Proper Interval Graphs

# Interval Graphs

- **Interval Graph:** A graph $G = (V, E)$ is said to be an **interval graph** if there exists a finite family $F$ of intervals in a linearly ordered set ( like the real line) and there is a one-to-one correspondence $f : V(G) \to F$ such that $uv \in E(G)$ if and only if $f(u) \cap f(v) \neq \emptyset$.

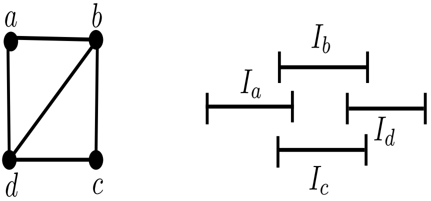



Figure: Example of an Interval Graph and the Interval representation

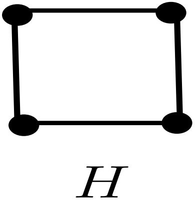



Figure: Example of not an Interval Graph

# Proper Interval Graphs

- **Proper Interval Graph:** If no interval of $F$ properly contains another interval of $F$ set-theoretically, then $G$ is called a **proper interval graph**.

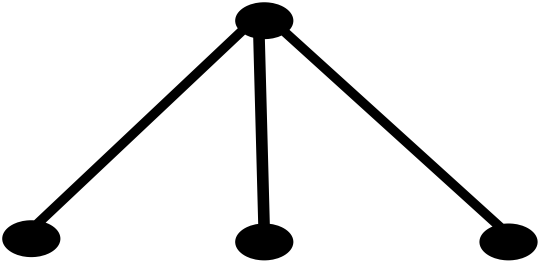


Figure: Example of not a proper interval Graph

# unit Interval Graphs

- **Unit Interval Graph:** If all the intervals in $F$ are unit length, then $G$ is called a **unit interval graph**.

# Remarks

- **Note:** proper interval graphs are same as unit interval graphs
- **Remark**: Does the definitions of Interval ( unit interval ) graphs depend on types of Intervals? ( closed: [a,b], open: (a,b), open-closed: (a,b], closed-open: (a,b], Mixed: )
- No for Interval Graphs. Yes for unit interval graphs.
- We consider closed Intervals.

# Interval Graphs are Subclass of Chordal Graphs

## Definition

A graph $G = (V, E)$ is said to be a **chordal graph** if every cycle in $G$ of length at least four has a chord, that is, an edge joining two non-consecutive vertices of the cycle.

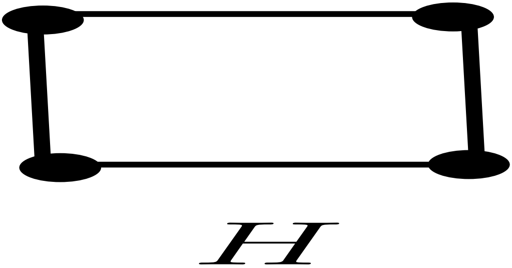



Figure: Example of not a chordal Graph

# Maximal Cliques

## Definition

- **Clique:** A subset $C \subseteq V(G)$ of a graph $G = (V, E)$ is called a **clique** if the induced subgraph $G[C]$, that is, $G[C] = (C, E')$, where $E' = \{xy | x, y \in C$ and $xy \in E(G)\}$, is a complete subgraph of $G$.
- **Maximal Clique:** If $C$ is a clique of $G$ and no proper superset of $C$ is a clique, then $C$ is called a **maximal clique** of $G$.

# Known Characterizations of Interval Graphs

**Theorem**

*[1] An undirected graph $G$ is an interval graph if and only if $C(G)$, the set of maximal cliques of $G$ can be linearly ordered such that for each vertex $x$ of $G$, $C_x(G)$ the maximal cliques containing $x$ occur consecutively.*

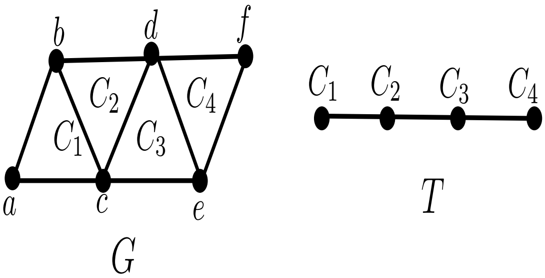



Figure: Example of an interval Graph and its clique tree

# Known Characterizations of Interval Graphs

**Theorem (Clique Tree Theorem)**

*A graph $G$ is an interval graph if and only if there exists a path $T$ such that $V(T) = C(G)$ and for every vertex $v$ of $G$, $T[C_v(G)]$ is a subpath of $T$.*

The path $T$ satisfying Clique Tree Theorem is called an **interval clique tree** of the interval graph $G$.

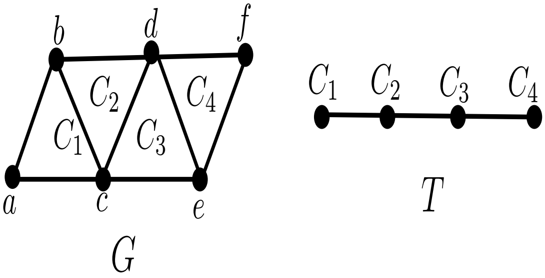



Figure: Example of an interval Graph and its clique tree

# Separated Subgraphs

## Definition

- **Separated Subgraphs:** If $G \setminus C$ is disconnected by a maximal clique $C$ into components $H_i$=($V_i$,$E_i$), $1 \leq i \leq r$, $r \geq 2$, then $C$ is said to be a **separating clique** and $G_i = G[V_i \cup C]$, $1 \leq i \leq r$ is said to be a **separated graph** of $G$ with respect to $C$.
- Monma and Wei [4] (JCT Ser B 1986) have characterized various subclasses of chordal graphs in terms of their separated subgraphs. The characterizations so obtained are called **separator theorems**. Separator theorems play an important role in designing algorithms in subclasses of chordal graphs.

# Separated Subgraphs

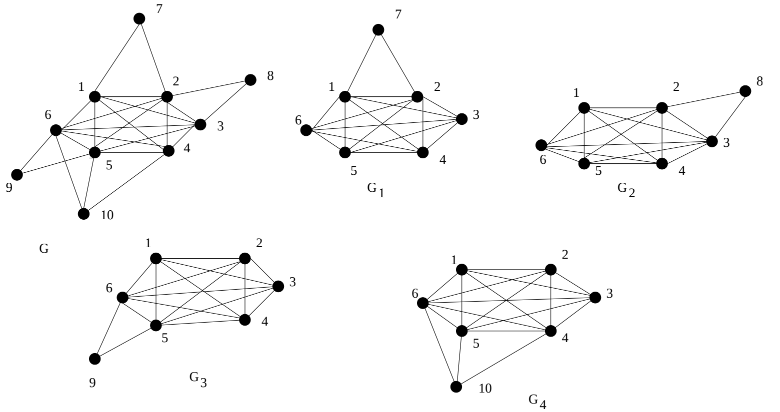



Figure: Example of the separation of G into $G_1, G_2, G_3$, and $G_4$ by $C = \{1, 2, 3, 4, 5, 6\}$

## Definition

Let G be a chordal graph, and $G_i$, $1 \leq i \leq r$, $r \geq 2$ be the separated graphs of G with respect to some separating clique $C$ of $G$. Cliques which intersect $C$ but not equal to $C$ are called **relevant cliques** with respect to $C$. For a separated graph $G_i$, let $W(G_i) = \{v \in C|$ there is a vertex $w \in (V(G_i) \setminus C)$ such that $vw \in E(G_i)\}$. Relevant cliques of $G_i$ which contain $W(G_i)$ are called **principal cliques** of $G_i$.

The existence of a principal clique of any separated graph of a chordal graph is assured by the following result due to Panda et al.[5]

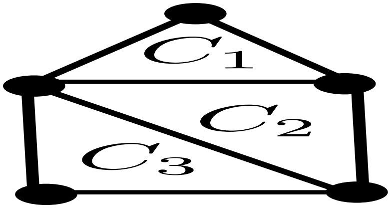



Figure: Relevant clique and principal clique

# Relation between relevant Cliques

In the following definitions, only relevant cliques are considered.

**Definition**

Let $C_1$ and $C_2$ be two cliques of $G$. We say

(1) $C_1$ and $C_2$ are **unattached**, denoted $C_1|C_2$, if $C_1 \cap C \cap C_2 = \emptyset$; otherwise, they are attached,

(2) $C_1$ **dominates** $C_2$, denoted $C_1 \geq C_2$, if $C_1 \cap C \supseteq C_2 \cap C$,

(3) $C_1$ **properly dominates** $C_2$, denoted $C_1 > C_2$, if $C_1 \cap C \supset C_2 \cap C$,

(4) $C_1$ and $C_2$ are **congruent,** denoted $C_1 \sim C_2$ , if they are attached and $C_1 \cap C = C_2 \cap C$, and

(5) $C_1$ and $C_2$ are **antipodal,** denoted $C_1 \iff C_2$, if they are attached and neither dominates the other.

# Relations between Separated Subgraphs

## Definition

Let $G_1$ and $G_2$ be two separated graphs of $G$ with respect to $C$. We say

1. $G_1$ and $G_2$ are **unattached**, denoted $G_1 \mid G_2$, if $C_1 \mid C_2$ for every clique $C_1$ in $G_1$ and for every clique $C_2$ in $G_2$; otherwise, they are attached,
2. $G_1$ dominates $G_2$, denoted $G_1 \geq G_2$, if they are attached and for every clique $C_1$ in $G_1$, $C_1 \geq C_2$ for all cliques $C_2$ in $G_2$ or $C_1 \mid C_2$ for all cliques $C_2$ in $G_2$,
3. $G_1$ properly dominates $G_2$, denoted $G_1 > G_2$ , if $G_1 \geq G_2$ but not $G_2 \geq G_1$,
4. $G_1$ and $G_2$ are congruent, denoted $G_1 \sim G_2$, if $G_1$ dominates $G_2$ and $G_2$ dominates $G_1$; in this case, $C_1 \sim C_2$ for all $C_1$ in $G_1$ and for all $C_2$ in $G_2$, and
5. $G_1$ and $G_2$ are **antipodal**, denoted $G_1 \Longleftrightarrow G_2$, if they are attached and neither dominates the other.

A separated graph $G_i$ with respect to $C$ is said to be **strictly relevant** if it has no nonrelevant cliques. Let $X(G_i)$ be the set of cliques of $G_i$ excluding $C$. Then, $\pi(V_i) = T[X(G_i)]$. Let $\pi(v)$ denote the path ( consisting of vertices) in $T$ corresponding to $v \in V$.

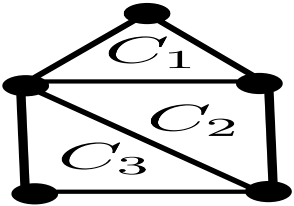



Figure: Strictly relevant separated graph

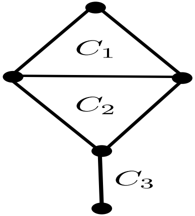



Figure: not a strictly relevant graph

# Separator Theorem for Interval Graphs

**Theorem (Separator Theorem)**

*Let $G_1, G_2 \ldots, G_r$, $r > 1$ be the separated graphs of a chordal graph $G$ with respect to a separating clique $C$. Then $G$ is an interval graph if and only if*

(i) *each $G_i$ is an interval graph,*

(ii) *the set of separated graphs can be $2$-colored such that no two antipodal graphs receive the same color,*

(iii) *in each color class no two relevant cliques are unattached, and*

(iv) *in each color class all separated graphs, except possibly one, are strictly relevant. The exceptional graph in (iv), should it exist, must be dominated by every separated graphs of like color.*

# Sketch of the Proof

Key: Uses the following theorem

**Theorem**

*A graph $G$ is an interval graph if and only if there exists a path $T$ such that $V(T) = C(G)$ and for every vertex $v$ of $G$, $T[C_v(G)]$ is a subpath of $T$.*

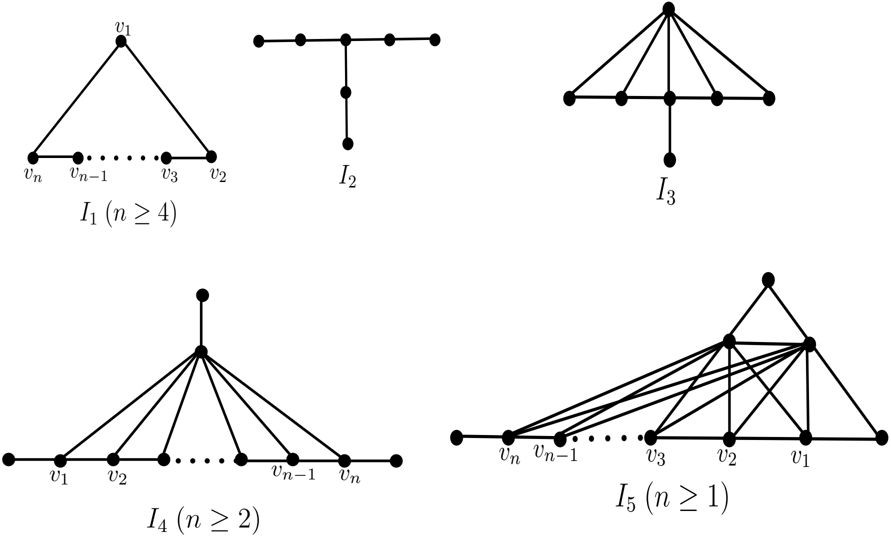
Result:
Let C separate G into separated graphs $G_1, G_2, \ldots, G_r$, $r \geq 2$.

**Lemma**

*If G is an interval graph, then each $G_i$ is an interval graph with a clique tree $T_i$ having C as an end vertex.*

# Forbidden Subgraph Characterization

- A graph class admits forbiddeb subgraph characterization iff it is closed under vertex induced subgraph.
- Interval graphs is closed under vertex induced subgraphs. Hence admits forbidden subgraph characterization.
- Lekkerkerker and Boland, (Representation of a finite graphs by a set of interval on a real line, Fundam. Math 51 (1962) 45-64), obtained the forbidden subgraphs for Interval graphs.
- **The forbiden subgraph characterization for interval graph can be obtained from our separator Theorem.**




Forbidden Subgraphs of Interval Graphs

**Theorem**

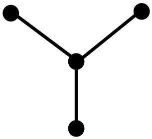
*[Separator Theorem] Let $G_1, G_2 \ldots, G_r$, $r > 1$ be the separated graphs of a chordal graph $G$ with respect to a separating clique $C$. Then $G$ is a proper interval graph if and only if*

(i) $r = 2$.

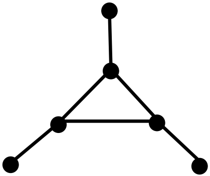
(ii) *Each $G_i$ is a proper interval graph,*

(iii) *If $W(G_1) \cap W(G_2) \neq \emptyset$, then $W(G_1) \cup W(G_2) = C$, and there is exactly one clique $C_i$ in $G_i$ intersecting $W(G_1) \cap W(G_2)$, $i = 1, 2$, and*

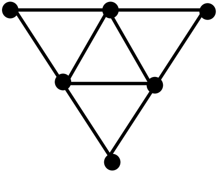
# Forbidden Subgraphs for proper Interval Graphs



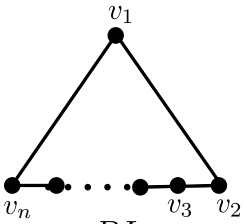
$PI_1$



$PI_2$



$PI_3$




$PI_4$

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# Thank you