Probabilistic arguments in graph coloring

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Graph Colouring

- A proper vertex k-colouring of $G = (V, E)$ is a labeled partition $V = V_1 \cup \ldots \cup V_k$ such that each V_i is an independent set in G.
- The *chromatic number* $\chi(G)$ is the minimum k such that G admits a proper vertex k-colouring.
- A proper edge k-colouring of $G = (V, E)$ is a labeled partition $E = E_1 \cup \ldots \cup E_k$ such that each E_i is a matching in G.
- The *chromatic index* $\chi'(G)$ is the minimum k such that G admits a proper edge k-colouring.

Immediate bounds on $\chi(G)$ and $\chi'(G)$

- $\delta(G)$ – minimum degree of G. $\Delta(G)$ – maximum degree of G.
- $\chi(G) \leq \Delta(G) + 1$.
- $\chi(G) \leq d(G) + 1$. $d(G)$ – degeneracy of G.
- degeneracy of $G = \max\{\delta(H) : H \subseteq G\}$. It is also the minimum value of the maximum out-degree of any acyclic orientation of G.
- planar graphs are 4-colorable and forests are 2-colorable.
- $\chi(G) \leq c\Delta/\log\Delta$ if G is triangle-free.
- $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ due to Vizing.

Complexity of $\chi(G)$ and $\chi'(G)$

- A $d(G) + 1$-coloring can be computed in linear time.
- An optimal $\chi(G)$-coloring can be computed efficiently for bipartite, chordal and perfect graphs.
- NP-Complete to determine whether $\chi(G) \leq 3$ even if G is a 4-regular planar graph.
- For arbitrary G, $\chi(G)$ is notoriously hard to approximate. Not approximable within a ratio of $O(n^{\delta})$ for any fixed $\delta < 1$, unless $NP = ZPP$.
- NP-Complete to determine whether $\chi'(G) \leq 3$ even if G is a 3-regular graph.
- $\chi'(G) \in \{\Delta, \Delta + 1\}$ always but $\chi(G)$ can vary widely with respect to Δ.

acyclic vertex colorings

- An acyclic vertex k-colouring of $G = (V, E)$ is a labeled partition $V = V_1 \cup \ldots \cup V_k$ such that (i) each V_i is independent, (ii) for every $i \neq j$, $G[V_i \cup V_j]$ is acyclic, i.e., no 2-colored cycle.
- The *acyclic chromatic number* $a(G)$ is the minimum k such that G admits an acyclic vertex k-colouring.
- $f_a(\Delta) = \max\{a(G) : \Delta = \Delta(G)\}$.
- It can be shown that $f_a(\Delta) = O(\Delta^{4/3})$. Also, known that $f_a(\Delta) = \Omega(\Delta^{4/3}/(\log \Delta)^{1/3})$. (Alon, McDiarmid and Reed, 1991). By probabilistic arguments.

acyclic edge colorings

- An acyclic edge k-colouring of $G = (V, E)$ is a labeled partition $E = M_1 \cup \ldots \cup M_k$ such that (i) each M_i is a matching, (ii) for every $i \neq j$, $M_i \cup M_j$ is acyclic, i.e., no 2-colored cycle.
- The *acyclic chromatic index* $a'(G)$ is the minimum k such that G admits an acyclic edge k-colouring.
- $f_{a'}(\Delta) = \max\{a'(G) : \Delta = \Delta(G)\}$.
- It is known that $f_{a'}(\Delta) \leq 64\Delta$. (Alon, McDiarmid and Reed, 1991). By probabilistic arguments.
- Conjecture : $a'(G) \leq \Delta(G) + 2$ for every G.

star vertex colorings

- A star vertex k-colouring of $G = (V, E)$ is a labeled partition $V = V_1 \cup \ldots \cup V_k$ such that (i) each V_i is independent, (ii) for every $i \neq j$, $G[V_i \cup V_j]$ is a collection of vertex disjoint stars, i.e., no 2-colored path on 4 vertices.
- The *star chromatic number* $\chi_s(G)$ is the minimum k such that G admits a star vertex k-colouring.
- $f_s(\Delta) = \max\{\chi_s(G) : \Delta = \Delta(G)\}$.
- It is known that $f_s(\Delta) = O(\Delta^{3/2})$. Also, known that $f_s(\Delta) = \Omega(\Delta^{3/2}/\sqrt{\log \Delta})$. (Fertin, Raspaud and Reed, 2004). By probabilistic arguments.

Applications

- acyclic vertex coloring models the optimal partitioning problem arising in a substitution based computation of a Hessian matrix.
- star vertex coloring models the optimal partitioning problem arising in a direct computation of a Hessian matrix.
- distace-2 coloring is closely related to the span of a radio-coloring of a graph and arises in mobile communication. Also, models the optimal partitioning problem arising in a direct computation of a Hessian matrix.

List Colorings :

- Given $G = (V, E)$, $\mathcal{L} = \{L_u : u \in V\}$.
- A $\mathcal{L}$-coloring of G is a map $f(u) \in L_u$ (for every u) such that $\{f(u)\}_u$ is a proper vertex coloring of G.
- Choice number $ch(G) =$ minimum k such that G admits a $\mathcal{L}$-coloring for every $\mathcal{L}$ satisfying $|L_u| \geq k$ for every u.
- $\chi(G) \leq ch(G)$ always.
- $\exists f : \mathcal{N} \to \mathcal{N}$ such that $ch(G) \leq f(\chi(G))$ for every G ?
- Answer is NO. Example : $ch(K_{n,n}) = \Theta(\ln n)$.
- $ch(G) \leq c\chi(\ln n)$ for every G with $n = |V(G)|$. By probabilistic arguments.
- $ch(G) \leq d(G) + 1$ always.
- $\exists f : \mathcal{N} \to \mathcal{N}$ such that $ch(G) \geq g(d(G))$ for every G ? Answer is YES. By probabilistic arguments. Noga Alon.

Oriented colorings

- an oriented graph $\vec{G}$ is an orientation of some undirected G.
- A $\vec{H}$-coloring of $\vec{G}$ is a homomorphism $h : V(\vec{G}) \to V(\vec{H})$ satisfying $(u, v) \in E(\vec{G}) \implies (h(u), h(v)) \in E(\vec{H})$.
- oriented chromatic number $\chi_o(\vec{G})$ = minimum k such that $\vec{G}$ is $\vec{H}$-colorable for some $\vec{H}$ on k vertices.
- oriented chromatic number $\chi_o(G) = \max_{\vec{G}}\{\chi_o(\vec{G})\}$.
- $d = \Delta(G)$. k = degeneracy of G.
- $\chi_o(G) \le d^3(6d)^d$ for any G (structural arguments, by CRS).
- $\chi_o(G) \le 2d^2 2^d$ for any G (prob. arguments, Kostochka, Sopena, Zhu).
- $\chi_o(G) \le 16kd2^k$ for any G (prob. arguments, Aravind and CRS).

Probabilistic arguments

- To prove : every G is colorable (in some fashion) using k colors.
- Choose a random coloring $\{f(u)\}_u$.
- Prove that $f(.)$ is a coloring of desired type with positive probability.
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- To prove : G is not always colorable (in some fashion) using k colors.
- Choose a random graph G.
- Prove that G is not colorable with k colors with positive probability.
- Use : tools and results from probability theory.

Probabilistic arguments in general

- Ω-Universe.
- To prove : there exists a $x \in \Omega$ satisfying a property P.
-
- Method : Fix a probability space (Ω, μ).
- Choose a random $\omega \in \Omega$.
- Establish that $\mathbf{Pr}(\omega \text{ satisfies } P) > 0$.
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- Deduce the desired conclusion.
- Use : tools and results from probability theory.

Existence of a subgraph with large cochromatic number

- cochromatic numbre $z(G)$ is the minimum k such that V can be partitioned into $V = V_1 \cup \ldots \cup V_k$ such that each $G[V_i]$ is either a clique or an independent set.
- $z(G) \leq \chi(G)$ always but can vary widely.
- Example : $z(K_n) = 1$ but $\chi(K_n) = n$.
- Possible to have $H \subseteq G$ with $Z(H) > Z(G)$.
- $Z^*(G) = \max_{H \subseteq G} Z(H)$.
- Alon, Krivelevich, Sudakov - JGT-1997.
- For any G, $Z^*(G) \geq \frac{t}{4(\log_2 t)} \cdot [1 - o(1)]$ where $t = \chi(G)$.
- Proof by probabilistic arguments.
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Proof of Existence of a subgraph

- Claim 1 : Either $z(G) \geq \frac{t}{\ln t}$ or $\exists$ a subgraph $G_1 = (V_1, E_1)$ satisfying $|V_1| \leq t^2$ and $\chi(G_1) \geq t \cdot [1 - o(1)]$.
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- Proof : Suppose $z(G) < \frac{t}{\ln t}$.
- $V = (I_1 \cup \ldots \cup I_k) \cup (C_1 \cup \ldots \cup C_l)$. $z(G) = k + l$.
- $V_1 = C_1 \cup \ldots \cup C_l$. $|V_1| < t^2/(\ln t)$. $G_1 = G[V_1]$.
- $t \leq \chi(G_1) + k \leq \chi(G_1) + \frac{t}{\ln t}$.
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- H - uniformly chosen random subgraph of G_1.
- Claim 2 : $z(H) \geq \left(\frac{1}{4} - o(1)\right) \cdot \frac{t}{\log_2 t}$ almost surely.
- Establishes the AKS theorem. the bound is nearly tight
- $z^*(G) \leq (2 + o(1)) \cdot \frac{n}{\log_2 n}$ for every G on n vertices.
- For $G = K_t$, the lower and upper bounds are within a constant multiplicative factor.

Proof of Claim 2

- Probability that H has a clique of size at most $4(\log_2 t)$ is :
- $\quad \leq \binom{t^2}{4(\log_2 t)} \cdot 2^{-\binom{4(\log_2 t)}{2}} = o(1).$
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- Probability that $\exists U \subseteq V_1 : \delta(G_1[U]) \geq 4(\log_2 t)$ and $H[U])$ is independent is :
- $\quad \sum_{k \geq 4(\log_2 t)} \binom{t^2}{k} \cdot 2^{-2k(\log_2 t)} = o(1).$
-
- with probability $1 - o(1)$, for every U such that $H[U]$ is independent, we have $\chi(G_1[U]) \leq 4(\log_2 t)$.
- $t[1 - o(1)] \leq \chi(G_1) \leq z(H) \cdot 4(\log_2 t)$.
- almost surely, $z(H) \geq \left(\frac{1}{4} - o(1)\right) \cdot \frac{t}{\log_2 t}$.

Proof of $\chi_o(G) \leq 16kd2^k$

- Due to NRA and CRS based on the one due to Kostochka, Sopena and Zhu.
- Claim : $\exists$ a tournament $T = (U, A)$ on $t = 16kd2^k$ vertices satisfying : for every ordered sequence $\vec{x} = (x_1, \ldots, x_i)$ of at most k vertices and for every tuple $\vec{a} \in \{IN, OUT\}^i$, there are at least $kd + 1$ other vertices in U each having their orientations into $(x_1, \ldots, x_i)$ governed by a.
-
- Fix an k-degenerate ordering $(v_n, \ldots, v_2, v_1)$ of V.
- Color the vertices in the reverse order with colors $f(v_i) \in U$ subject to maintaining always :
- the partial coloring $f(v_1), \ldots, f(v_i)$ is always a T-coloring.
- For each v_j with $j > i$, all neighbors of v_j in $\{v_1, \ldots, v_i\}$ are distinctly colored.
- one can extend the coloring iteratively using Claim.

Proof of Claim

- Choose uniformly at random a tournament T on $t = 16kd2^k$ vertices.
- Given $\vec{x}$ and $\vec{a}$, $X_{\vec{x},\vec{a}} =$ number of vertices in U having their orientations into $\vec{x}$ governed by $\vec{a}$.
- $X_{\vec{x},\vec{a}}$ is the sum of $t - i$ iid indicator variables.
- $E[X_{\vec{x},\vec{a}}] = (t-i)2^{-i} \geq (t-d)2^{-d}$.
- $\mathbf{Pr}(X_{\vec{x},\vec{a}} \leq kd) \leq e^{-(3.75)kd}$.
- $\mathbf{Pr}(\exists \vec{x}, \vec{a} : X_{\vec{x},\vec{a}} \leq kd) \leq d \cdot \binom{t}{d} \cdot 2^d \cdot e^{-(3.75)kd} < 1$.
- $\mathbf{Pr}(\exists \vec{x}, \vec{a} : X_{\vec{x},\vec{a}} \geq kd + 1) > 0$.

Lovasz Local Lemma

- $\mathcal{A} = \{A_1, \ldots, A_m\}$ – bad events.
- For each i, A_i is mutually independent of all but those in $\mathcal{F}_i \subseteq \mathcal{A} \setminus \{A_i\}$.
- Suppose $\{y_i : 0 \leq y_i < 1\}$ are reals such that
- for each i, $\mathbf{Pr}(A_i) \leq y_i \prod_{A_j \in \mathcal{F}_i}(1 - y_j)$.

- Then, $\mathbf{Pr}\left(\wedge_i \overline{A_i}\right) \;\geq\; \prod_i (1 - y_i) \;>\; 0.$

- Need to choose suitable bad events.
- Need to properly choose of y_i's.

Bounds on acyclic chromatic numbers

- Weaker bound $f_a(d) \leq 16d^{3/2}$ is presented.
- $x = 16d^{3/2}$ - number of colors.
- Choose $f : V \to [x]$ uniformly randomly.
- Type 1 : For every $uv \in E$, E_{uv} denotes $f(u) = f(v)$.
- Type $(2, k)$: For every even cycle C of length $2k$, $E_{C,2k}$ denotes C is properly bicolored.
- $\mathbf{Pr}(E_{uv}) = \frac{1}{x}$; $\mathbf{Pr}(E_{C,2k}) \leq 1/x^{2k-2}$.
- Each event on r vertices is independent of all but at most rd events of Type 1 and at most rd^{2k-1} events of Type $(2, k)$.
- Choose $y_E = \frac{2}{x}$ if E is of Type 1 and $y_E = \left(\frac{2}{x}\right)^{2k-2}$ if E is of Type $(2, k)$.
- inequalities can be verified to be satisfied.
- Conclusion : G is acyclically colorable using x colors.

Stronger and tight bounds on acyclic chromatic numbers

- Due to Alon, McDiarmid and Reed, RSA'91.
- a non-adjacent pair (u, v) is a special pair if they share more than $d^{2/3}$ common neighbors.
- $x = Cd^{4/3}$ - number of colors.
- Choose $f : V \to [x]$ uniformly randomly.
- Type 1 : For every $uv \in E$, E_{uv} denotes $f(u) = f(v)$.
- Type 2 : For every path $P = (u_0, u_1, u_2, u_3, u_4)$ of length 4, E_P denotes $f(u_0) = f(u_2) = f(u_4)$ and $f(u_1) = f(u_3)$.
- Type 3 : For every induced cycle $C = (u_1, u_2, u_3, u_4, u_1)$ of length 4 with neither (v_1, v_3) nor (v_2, v_4) being a special pair, let E_C denote $f(u_1) = f(u_3)$ and $f(u_2) = f(u_4)$.
- Type 4 : for every special pair (u, v), let $E_{u,v}$ denote $f(u) = f(v)$.
- absence of events of each of these types implies f is a proper cyclic coloring of G.
- applying LLL, one can deduce that $a(G) \leq Cd^{4/3}$.

Generalization - forbidden subgraph colorings

- j – positive integer.
- $\mathcal{F}$ – family of connected j-colorable graphs H more than j vertices each.
- $(j, \mathcal{F})$-subgraph coloring :
- a partition $V = V_1 \cup \ldots \cup V_s$ such that
  1. Each V_i is independent.
  2. union of *any* j color classes induces a subgraph which is free of any member of $\mathcal{F}$.
- $(j, \mathcal{F})$-chromatic number $\chi_{j,\mathcal{F}}(G) =$ minimum number s of colors that guarantees the existence of such a coloring.
- $\chi_{j,\mathcal{F}}(G) \leq Cd^{\frac{k-1}{k-j}}$ for every G;
- $k = \min\{|V(H)| : H \in \mathcal{F}\}$.
- Due to Aravind and CRS, JGT-2011.

Bounds on forbidden subgraph colorings

- Bounds on (j, k)-colorings :
  - j, k – positive integers with $j \leq k$.
  - G – arbitrary graph with $d = \Delta(G)$.
  - $C = C(j, k)$.
  - $\chi_{j,k}^{con}(G) \leq \lceil C d^{\frac{k}{k+1-j}} \rceil$.

- Bounds on $(j, \mathcal{F})$-subgraph colorings :
  - j and $\mathcal{F}$ as defined before.
  - $k = \min\{|V(H)| : H \in \mathcal{F}\}$; $j \leq k - 1$.
  - G – arbitrary graph with $d = \Delta(G)$; $D = D(j, k-1)$.
  - $\chi_{j,\mathcal{F}}(G) \leq \lceil D d^{\frac{k-1}{k-j}} \rceil = \lceil D d^{1+\frac{j-1}{k-j}} \rceil$.

Specializations

- star chromatic number $\chi_s(G)$:

  $j = 2$, $\mathcal{F} = \{P_4\}$, $k = 4$;

  $\chi_s(G) = O(d^{3/2}) =$ the bound mentioned before.

- distance-2 chromatic number $\chi_{d2}(G)$:

  $j = 2$, $\mathcal{F} = \{P_3\}$, $k = 3$;

  $\chi_s(G) = O(d^2) =$ the bound mentioned before.

- acyclic chromatic number $\chi_s(G)$:

  $j = 2$, $\mathcal{F} = \{C_4, C_6, \ldots\}$, $k = 4$;

  $\chi_s(G) = O(d^{3/2}) =$ weaker than the bound mentioned before.

Nearly tight bounds for the case $j = 2$

- Due to Aravind and CRS - EJC-2013.
- Tight bounds on $(2, \mathcal{F})$-subgraph colorings :
  - $\mathcal{F}$ – family of connected bipartite graphs H on 3 or more vertices.
  - $m = \min\{|E(H)| : H \in \mathcal{F}\}$;
  - G – arbitrary graph with $d = \Delta(G)$;
  - $C = C(\mathcal{F})$.
  - $\chi_{2,\mathcal{F}}(G) \leq \lceil Cd^{\frac{m}{m-1}} \rceil = \lceil Cd^{1+\frac{1}{m-1}} \rceil$.
- Acyclic vertex coloring $= (2, \{C_4, C_6, \ldots\})$-subgraph coloring.
- $a(d) = O(d^{4/3}) =$ the bound mentioned before.
- $\chi_b^{frug}(G) =$ least k such that G can be properly k-colored with each u having at most b neighbors colored the same.
- $\chi_b^{frug}(d) = \max\{\chi_b^{frug}(G) : \Delta(G) = d\} = O(d^{(b+1)/b})$.

How good are the bounds for the case $j = 2$

- Tightness of bounds on $(2, \mathcal{F})$-subgraph colorings :
- Due to AS (EJC-2013);
  - $\mathcal{F}$ – family of connected bipartite graphs H on 3 or more vertices.
  - $m = \min\{|E(H)| : H \in \mathcal{F}\}$;
  - For infinitely many values of d,
  - there are graphs G having maximum degree d and
  - $\chi_{2,\mathcal{F}}(G) = \Omega\left(\frac{d^{\frac{m}{m-1}}}{(\log d)^{1/(m-1)}}\right)$.
  - Proof based on analyzing $G(n, p)$ for suitably chosen p.
- Acyclic vertex coloring $f_a(d) = \Omega\left(\frac{d^{4/3}}{(\log d)^{1/3}}\right)$ = the lower bound mentioned before.
- star vertex coloring $f_s(d) = \Omega\left(\frac{d^{3/2}}{\sqrt{\log d}}\right)$ = the lower bound mentioned before.

How good are the bounds for $\chi_2^{frug}(G)$

- Proof of : this bound is tight upto a $\sqrt{\ln d}$ factor.
- Consider $G \in \mathcal{G}(n,p)$ with $p = 6\left(\frac{\ln n}{n}\right)^{1/3}$.
- expected degree $\mu \approx 6\left(n^{2/3}(\ln n)^{1/3}\right)$.
- $\mathbf{Pr}(\frac{\mu}{2} \leq d \leq 2\mu) \to 1$ as $n \to \infty$.
- Fix any partition $V = V_1 \cup \ldots V_s$ with $s \leq n/3$.
- Remove at most two vertices from each V_i and further split each part to
- Get a collection $(W_1, \ldots, W_r)$ $(r \geq n/9)$ with $|W_i| = 3$ for each i :
- $G[W_i \cup W_j]$ has maximum degree less than 3 for every $i \neq j$.
- $\mathbf{Pr}(\chi_2^{frug}(G) \leq n/3) \leq n^n(1-p^3)^{\binom{n/9}{2}} = o(1)$.
- with prob. approaching 1, $\chi_2^{frug}(G) > n/3$ and $\Delta(G) \leq 2\mu$.
- for infinitely many values of d, there are G with $\chi_2^{frug}(G) \geq C\frac{d^{3/2}}{(\ln d)^{1/2}}$.

Further Generalization - forbidding several families

- $\mathcal{P} = \{(j_1, \mathcal{F}_1), \ldots, (j_l, \mathcal{F}_l)\}$.
- $\mathcal{P}$-subgraph coloring :
- a partition $V = V_1 \cup \ldots \cup V_s$ such that
  1. Each V_i is independent.
  2. for each $i \leq l$, union of *any* j_i color classes induces a subgraph which is free of any member of $\mathcal{F}_i$.

- $k_i = \min\{|V(H)| : H \in \mathcal{F}_i\}$; $D = D(\mathcal{P})$.
- $\mathcal{P}$-forbidden chromatic number $\chi_{\mathcal{P}}(G)$ = minimum number s of colors that guarantees the existence of such a coloring.
- $\chi_{\mathcal{P}}(G) \leq \lceil Dd^{\max_i \frac{k_i - 1}{k_i - j_i}} \rceil$.

Generalization - forbidden subgraph edge colorings

- j – positive integer.
- $\mathcal{F}$ – family of connected graphs H such that $\chi'(H) \leq j$ and $j < |E(H)|$.

- $(j, \mathcal{F})$-subgraph edge coloring :
- a partition $E = E_1 \cup \ldots \cup E_s$ such that
  (1) Each E_i is a matching.
  (2) union of *any* j color classes forms a subgraph which is free of any member of $\mathcal{F}$.
- if $j < |E(H)|$, then there is always one such coloring.
- $(j, \mathcal{F})$-chromatic index $\chi'_{j,\mathcal{F}}(G)$ = minimum number s of colors that guarantees the existence of such an edge coloring.

Bounds on forbidden subgraph edge colorings

- j and $\mathcal{F}$ as defined before.
- $\theta(j, \mathcal{F}) = \max_{H \in \mathcal{F}} \frac{|V(H)|-2}{|E(H)|-j}$.
- G – arbitrary graph with $d = \Delta(G)$; $C = C(\mathcal{F})$.
- $\chi'_{j,\mathcal{F}}(G) \leq \lceil Cd^{\max\{\theta,1\}} \rceil$.

- Corollary :
- j and $\mathcal{F}$ as before.
- $D = D(\mathcal{F}) = \min\{|E(H)| - |V(H)| : H \in \mathcal{F}\}$.
- Suppose $j \leq D + 2$. Then, for $C = C(\mathcal{F})$,
- $\chi'_{j,\mathcal{F}}(G) \leq \lceil Cd \rceil$.

Specializations

- Given G with $d = \Delta(G)$, $O(d)$ colors suffice to
- ensure the existence of proper edge colorings such that
  - union of any 2 color classes is acyclic – acycic edge coloring.
  - union of any 3 color classes is outerplanar.
  - union of any 4 color classes is a partial 2-tree.
  - union of any 5 color classes is planar.
  - union of any $k+2$ color classes is a partial k-tree, provided $k \geq 2$.
  - union of any $(k^2+k+2)/2$ color classes is a k-colorable graph.
  - union of any $(k^2-k+2)/2$ color classes is a k-degenerate graph.
  - union of any $2g+3$ color classes is a graph of genus at most g.
- In fact, all of these requirements can be simultaneously met.

List chromatic numbers

- For any G, $ch(G) \leq c\chi(G)(\ln n)$.
- proved by looking at a simple random truncation of each list obtained by choosing uniformly at random a χ-partition of $\cup_u L_u$.
- suffices to show that each truncated list is non-empty and lists for adjacent vertices are disjoint with positive probability.
- Improved by CRS (CPC-2007)
- For any G, $\chi(G) \leq c\chi(G)\left(\ln\left(\frac{n}{\chi}\right) + 1\right)$.
- based on extending and generalizing the prob. proof arguments of
- Alon worked out for a very special class of graphs.
- The random pruning has to take into account the sizes of color classes of a fixed optimal coloring. Such a choice was obtained.
- The result can be extended to list versions of hereditary chromatic numbers, list versions of hypergraph chromatic numbers, etc.

Algorithmic issues

- **constructibility :** The recent breakthrough of Gabor and Tardos can be applied to get a randomized algorithm with an EXPECTED polynomial running time to produce a $(j, \mathcal{F})$-subgraph coloring matching the upper bounds mentioned.
- This, for example, produces the first poly time algorithms matching the bounds for such coloring notions as acyclic vertex coloring, acyclic edge coloring, etc.
- Focus on improved bounds on $(j, \mathcal{F})$-subgraph colorings for some special classes of graphs.
- Some specific questions in this direction are being pursued.

Thank You